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ON THE EXISTENCE OF INVARIANT MEASURES FOR PIECEWISE MONOTONIC TRANSFORMATIONS(1)

BY

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ABSTRACT. A class of piecewise continuous, piecewise C^1 transformations on the interval $[0, 1]$ is shown to have absolutely continuous invariant measures.

1. **Introduction.** The purpose of this note is to prove the existence of absolutely continuous invariant measures for a class of point-transformations of the unit interval [0, 1] into itself. Our main result is Theorem 1 which generalizes some previous results of A. Rényi [5], A. O. Gel′fond [2], W. Parry [4] and A. Lasota [3]. It gives, also, a positive answer to a conjecture of S. Ulam [7, p. 74]. Theorem 1 is stated for a piecewise monotonic function with a finite number of discontinuities but it can be easily extended to some piecewise monotonic functions with infinite number of discontinuities.

Our method is different from the methods of the above mentioned authors. Firstly we explore the fact that the Frobenius-Perron operator corresponding to the point-transformation under consideration has the property of sometimes shrinking the variation of the function. Secondly to prove the existence of invariant measures we use the abstract ergodic theorem which enables us to make our proofs constructive. The advantage of this method is that we do not require that our mappings be local homeomorphisms nor that they generate an exact endomorphism in the sense of Rohlin [6], a property that has been the typical requirement for previous work. §4 describes some extensions, including an extension to higher dimensions.

2. **Existence theorem.** Denote by $(L_1, \| \|)$ the space of all integrable functions defined on the interval [0, 1]. Lebesgue measure on [0, 1] will be denoted by m. Let τ: $[0, 1] \rightarrow [0, 1]$ be a measurable nonsingular transformation.

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"Nonsingularity" means that $m(\tau^{-1}(A)) = 0$ whenever $m(A) = 0$ for a measurable set A. Given τ we define the Frobenius-Perron operator P_τ: $L_1 \to L_1$ by the formula

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}([0,x])} f(s)\, ds.$$

It is well known that the operator P_τ is linear and continuous and satisfies the following conditions:

(a) P_τ is positive: $f \geq 0 \Rightarrow P_\tau f \geq 0$;

(b) P_τ preserves integrals

$$\int_0^1 P_\tau f\, dm = \int_0^1 f\, dm, \qquad f \in L_1;$$

(c) $P_{\tau^n} = P_\tau^n$ (τ^n denotes the nth iterate of τ);

(d) $P_\tau f = f$ if and only if the measure $d\mu = f\,dm$ is invariant under τ, that is $\mu(\tau^{-1}(A)) = \mu(A)$ for each measurable A.

A transformation τ: $[0, 1] \to R$ will be called *piecewise* C^2, if there exists a partition $0 = a_0 < a_1 < \cdots < a_p = 1$ of the unit interval such that for each integer i ($i = 1, \dots, p$) the restriction τ_i of τ to the open interval (a_{i-1}, a_i) is a C^2 function which can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^2 function. τ need not be continuous at the points a_i.

Theorem 1. *Let* τ: $[0, 1] \to [0, 1]$ *be a piecewise* C^2 *function such that* $\inf |\tau'| > 1$. *Then for any* $f \in L_1$ *the sequence*

$$\frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f$$

is convergent in norm to a function $f^* \in L_1$. *The limit function has the following properties*:

(1) $f \geq 0 \Rightarrow f^* \geq 0$.

(2) $\int_0^1 f^* dm = \int_0^1 f\, dm$.

(3) $P_\tau f^* = f^*$ *and consequently the measure* $d\mu^* = f^* dm$ *is invariant under* τ.

(4) *The function* f^* *is of bounded variation*; *moreover, there exists a constant* c *independent of the choice of initial* f *such that the variation of the limiting* f^* *satisfies the inequality*(2)

$$\bigvee_0^1 f^* \leq c\|f\|.$$

(2) Here and in what follows the symbol $\bigvee_a^b f$ as well as $\bigvee_{[a,b]} f$ denote the varition of f over the closed interval $[a, b]$.

We point out in Theorem 3 that it is sufficient to assume just that some iterate of τ satisfy the derivative condition.

Proof. Write $s = \inf|\tau'|$ and choose a number N such that $s^N > 2$. It is easy to see that the function $\phi = \tau^N$ is piecewise C^2. Denote by $b_0, \cdots, b_q$ the corresponding partition for ϕ. Writing ϕ_i for the corresponding C^2 functions we have

$$|\phi_i'(x)| \geq s^N, \quad x \in [b_{i-1}, b_i], \; i = 1, \cdots, q. \tag{5}$$

Computing the Frobenius-Perron operator for ϕ we obtain

$$P_\phi f(x) = \sum_{i=1}^{q} f(\psi_i(x))\sigma_i(x)\chi_i(x) \tag{6}$$

where $\psi_i = \phi_i^{-1}$, $\sigma_i(x) = |\psi_i'(x)|$ and χ_i is the characteristic function of the interval $J_i = \phi_i([b_{i-1}, b_i])$. From (5) it follows that

$$|\sigma_i(x)| \leq s^{-N}, \quad x \in J_i, \; i = 1, \cdots, q. \tag{7}$$

By its very definition the operator P_ϕ is defined as a mapping from L_1 into L_1 but the formula (6) enables us to consider P_ϕ as a map from the space of functions defined on $[0, 1]$ into itself.

Let f be a given function of bounded variation over $[0, 1]$. From (6) and (7) it follows that

$$\bigvee_0^1 P_\phi f \leq \sum_{i=1}^{q} \bigvee_{J_i} (f \circ \psi_i)\sigma_i + s^{-N} \sum_{i=1}^{q} (|f(b_{i-1})| + |f(b_i)|). \tag{8}$$

In order to evaluate the first sum we write

$$\begin{aligned}
\bigvee_{J_i} (f \circ \psi_i)\sigma_i &= \int_{J_i} |d((f \circ \psi_i)\sigma_i)| \\
&\leq \int_{J_i} |f \circ \psi_i||\sigma_i'|\, dm + \int_{J_i} \sigma_i |d(f \circ \psi_i)| \\
&\leq K \int_{J_i} |f \circ \psi_i|\sigma_i\, dm + s^{-N} \int_{J_i} |d(f \circ \psi_i)|
\end{aligned}$$

where $K = \max|\sigma_i'|/\min(\sigma_i)$. Changing the variables we obtain

$$\bigvee_{J_i} (f \circ \psi_i)\sigma_i \leq K \int_{b_{i-1}}^{b_i} |f|\, dm + s^{-N} \int_{b_{i-1}}^{b_i} |df|. \tag{9}$$

In order to evaluate the second term in (8) we write

$$|f(b_{i-1})| + |f(b_i)| \leq \bigvee_{b_{i-1}}^{b_i} f + 2d_i \tag{10}$$

where $d_i = \inf\{|f(x)| : x \in [b_{i-1}, b_i]\}$. On the other hand we have an obvious inequality

$$d_i \le b^{-1} \int_{b_{i-1}}^{b_i} |f|\, dm \tag{11}$$

where $b = \min_i (b_i - b_{i-1})$. From (10), (11) it follows that

$$\sum_{i=1}^{q} (|f(b_{i-1})| + |f(b_i)|) \le \bigvee_0^1 f + 2b^{-1}\|f\|. \tag{12}$$

Applying (12) and (9) to (8) we obtain $\bigvee_0^1 P_\phi f \le \alpha\|f\| + \beta \bigvee_0^1 f$ where $\alpha = (K+2b^{-1})$ and $\beta = 2s^{-N} < 1$.

Now, for the same function f, let us write $f_k = P_\tau^k f$. Since $P_\tau^N = P_\phi$ we have

$$\bigvee_0^1 f_{Nk} \le \alpha\|f_{N(k-1)}\| + \beta \bigvee_0^1 f_{N(k-1)} \le \alpha\|f\| + \beta \bigvee_0^1 f_{N(k-1)}$$

and consequently

$$\limsup_{k\to\infty} \bigvee_0^1 f_{Nk} \le \alpha(1-\beta)^{-1}\|f\|. \tag{13}$$

The last inequality and the condition $\|f_k\| \le \|f\|$ (which follows from (a) and (b)) prove that the set $C = \{f_{Nk}\}_{k=0}^{\infty}$ is relatively compact in L_1. Since $\{f_k\}_{k=0}^{\infty} \subset \bigcup_{k=0}^{N-1} P_\tau^k C$, the whole sequence $\{f_k\}_{k=0}^{\infty}$ is relatively compact, too. By Mazur's theorem the same is true for the sequence

$$\left\{\frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f\right\}. \tag{14}$$

The set of functions of bounded variation is dense in L_1. We have proved that for any such function f the sequence (14) is relatively compact. Therefore, we are in a position to use the Kakutani-Yosida Theorem (see [1, VIII.5.3]) which says that for any $f \in L_1$ the sequence (14) converges strongly to a function f^* which is invariant under P_τ. From (a) and (b) it follows that f^* satisfies (1) and (2). Therefore it remains only to prove (4). Since the operator P_τ is given by a formula analogous to (6) it is easy to derive the inequality $\bigvee_0^1 P_\tau f \le c_1 \bigvee_0^1 f + c_2\|f\|$ with some constants c_1 and c_2. This and (13) imply the inequality

$$\limsup_{k\to\infty} \bigvee_0^1 P_\tau^k f \le c\|f\|$$

(with a positive constant c) which is valid for any f with bounded variation. Consequently for any such f we have also

$$\limsup_{k\to\infty} \bigvee_0^1 \left(\frac{1}{n}\sum_{k=1}^{n-1} P_\tau^k f\right) \le c\|f\|.$$

Writing $Q = \lim_n (1/n)\sum_{k=1}^{n-1} P_\tau^k$ and using Helly's theorem we have $\bigvee_0^1 Qf \le c\|f\|$, for f of bounded variation. The operator Q is linear and contractive. We may therefore apply Helly's theorem once more to extend this inequality for the closure of the set of functions of bounded variation, that is to all of L_1. This finishes the proof.

3. **A counterexample.** Now we shall show that our assumption $\inf|\tau'| > 1$ is essential. Consider the transformation

$$\gamma(x) = \begin{cases} x/(1-x) & \text{for } 0 \le x < \tfrac12, \\ 2x-1 & \text{for } \tfrac12 \le x \le 1 \end{cases}$$

for which the assumption $|\gamma'(x)| > 1$ is violated only at $x = 0$. We are going to prove that for any $f \in L^1$ the sequence $P_\gamma^n f$ converges in measure to zero. Therefore the equation $P_\gamma f = f$ has only the trivial solution and there is no absolutely continuous nontrivial measure invariant under γ.

The proof will be given in a few steps. First we prove that for $f_0 \equiv 1$ the sequence $g_n(x) = xf_n(x)$, where $f_n = P_\gamma^n f_0$, converges to a constant c_0. Then using the condition $\|f_n\| = 1$ we derive easily that $c_0 = 0$, and consequently $f_n \to 0$. Finally by an approximation argument we may extend this result to an arbitrary sequence $P_\gamma^n f$ with $f \in L_1$.

The Frobenius-Perron operator P_γ may be written in the form

$$P_\gamma f(x) = \frac{1}{(1+x)^2} f\left(\frac{x}{1+x}\right) + \frac12 f\left(\frac12 + \frac{x}{2}\right).$$

Thus for g_n we have the following recursive formula:

$$g_{n+1}(x) = \frac{1}{1+x}\, g_n\left(\frac{x}{1+x}\right) + \frac{x}{1+x}\, g_n\left(\frac12 + \frac{x}{2}\right), \qquad g_0(x) \equiv x. \tag{15}$$

By an induction argument it is easy to check that $g_n' \ge 0$ for each n. Therefore all the functions g_n are positive and increasing. According to (15) we have

$$g_{n+1}(1) = \tfrac12 g_n(\tfrac12) + \tfrac12 g_n(1) \le g_n(1).$$

This proves the existence of a limit $\lim_n g_n(1) \overset{\text{df}}{=} c_0$. Write $z_0 = 1$ and $z_{k+1} = z_k/(1+z_k)$. According to (15) we obtain

$$g_{n+1}(z_k) = \frac{1}{1+z_k}\, g_n(z_{k+1}) + \frac{z_k}{1+z_k}\, g_n\left(\frac12 + \frac{z_k}{2}\right).$$

Fix k and suppose that $\lim_n g_n(x) = C_0$ for $z_k \le x \le 1$. (This is certainly true for $k = 0$.) Since $z_k \le \tfrac12 + \tfrac12 z_k$, we obtain at the limit as $n \to \infty$

$$C_0 = \frac{1}{1+z_k} \lim_n g_n(z_{k+1}) + \frac{z_k}{1+z_k} C_0.$$

Thus $\lim_n g_n(z_{k+1}) = C_0$. Since g_n are increasing, this proves that $\lim_n g_n(x) = C_0$ uniformly for all $x \in [z_{k+1}, 1]$. Therefore by an induction argument it follows that $\lim_n g_n(x) = C_0$ in any interval $[z_k, 1]$ and consequently, since $\lim_k z_k = 0$, we have $\lim_n g_n(x) = C_0$ for all $0 < x \leq 1$. Hence, $\lim_n f_n(x) = c_0/x$. We claim that $c_0 = 0$. If not there would exist $\epsilon > 0$ such that $\int_\epsilon^1 c_0/x\, dx > 1$ and

$$\lim_n \int_\epsilon^1 f_n(x)\, dx = \int_\epsilon^1 \frac{c_0}{x}\, dx > 1$$

which is impossible since $\|f_n\| = 1$ for each n. It can be easily proved by induction that each of the functions f_n is decreasing. Thus the convergence of f_n to zero is uniform on any interval $[\epsilon, 1]$ with $\epsilon > 0$.

Now let f be an arbitrary function. We may write $f = f^+ - f^-$ where $f^+ = \max(0, f)$ and $f^- = \max(0, -f)$. Given $\epsilon > 0$ consider a constant r such that

$$\int_0^1 (f^- - r)^+\, dm + \int_0^1 (f^+ - r)^+\, dm \leq \epsilon.$$

We have

$$\int_\epsilon^1 |P_\gamma^n f|\, dm = \int_\epsilon^1 P_\gamma^n f^+\, dm + \int_\epsilon^1 P_\gamma^n f^-\, dm$$

$$\leq 2\int_\epsilon^1 P_\gamma^n r\, dm + \int_\epsilon^1 P^n(f^+ - r)\, dm + \int_\epsilon^1 P^n(f^- - r)\, dm$$

$$\leq 2r \int_\epsilon^1 P_\gamma^n 1\, dm + \epsilon.$$

Since $P_\gamma^n 1$ converges on $[\epsilon, 1]$ uniformly to zero we have

$$\lim_n \int_\epsilon^1 |P_\gamma^n f|\, dm = 0 \qquad \text{for } \epsilon > 0$$

which proves that the sequence $P_\gamma^n f$ converges in measure to zero.

4. Final remarks. Now we want to discuss some extensions of our method to other transformations. First of all we may prove an analogue of Theorem 1 for piecewise C^2 transformations with a countable number of pieces.

Let $r_i\colon \Delta_i \to [0, 1]$ be a countable sequence of C^2 functions where Δ_i is a sequence of closed intervals such that $\Sigma_i\, m(\Delta_i) = 1$, $m([0, 1] - \bigcup_i \Delta_i) = 0$. The function r defined by the condition

$$r(x) = r_i(x), \qquad x \in \text{interior of } \Delta_i,$$

will be called countably piecewise C^2. Note that the values of r on the set $[0, 1] \setminus \bigcup_i \operatorname{int} \Delta_i$ are arbitrary.

Theorem 2. *Let τ be a countably piecewise C^2 function such that*

$$\inf|\tau'(x)| > 2, \qquad \sup|\tau''(x)| < \infty, \tag{16}$$

$$\tau_i(\Delta_i) = [0, 1] \quad \textit{except for a finite number of intervals.} \tag{17}$$

Then for each $f \in L_1$ the sequence $(1/n)\sum_{k=0}^{n-1} P_\tau^k f$ is convergent in norm to a function f^* which satisfies conditions (1), (2), (3) and (4).

The proof of Theorem 2 is basically the same as the proof of Theorem 1. Thus it can be omitted. Let us only note that the condition (17) is essential. In fact it is easy to construct a countably piecewise linear function with the slope $\tau' > 3$ such that $\inf_{\epsilon \le x \le 1-\epsilon} \tau(x) - x$ is a positive number for each ϵ in $(0, 1/2)$. (The graph of τ lies over the diagonal.) It can be proved by elementary calculation that for any such function τ and $f \in L_1$, $P_\tau^n f \to 0$ in measure as $n \to \infty$.

A close look at the proof of Theorem 1 shows that we have used only the fact that $\sup|(\tau^N)'| > 2$. Therefore, in fact, we have proved the following result.

Theorem 3. *Let τ: $[0, 1] \to [0, 1]$ be a piecewise C^2 function such that $\inf|(\tau^{n_0})'| > 1$ for a positive integer n_0. Then for any $f \in L_1$ the sequence $(1/n)\sum_{k=0}^{n-1} P_\tau^k f$ is convergent in norm to a function f^* which satisfies conditions* (1), (2) *and* (3). *If, in addition,* $\inf|\tau'| > 0$ *then condition* (4) *is also satisfied.*

Observe that in our counterexample the function γ has the property that $(\gamma^n)'_{x=0} = 0$ for each n. This is because the point $(0, \gamma(0))$ lies on the diagonal.

Our techniques can be easily used to obtain new proofs of known results in higher dimensions. See [8], [9], [10] for such results. In this case τ: $M \to M$ is assumed C^1 on a compact manifold M and the variation of a C^1 function f is defined as $\int_M |\text{grad} f(m)|\, dm$. Hence in this case we do not allow discontinuities in τ, or more generally if f is C^1 on $M \setminus \partial M$, we must make assumptions on τ guaranteeing $P_\tau(f)$ is C^1 on $M \setminus \partial M$. The techniques in [8], [9], [10] are quite different from the "bounded variation" approach of this paper.

The study of the functions τ described arose while investigating the design of more durable high speed oil well drilling bits. The invariant measure $f(x)\,dx$ describes the distribution of impacts on the surface of the bit. The durability and efficiency of the tool depends strongly on f. The first author is part of a team that has obtained patents in Poland for superior bits by slightly altering the bit shape to one with a better impact distribution f.

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