ON THE SUM OF TWO SOLID ALEXANDER HORNED SPHERES

BY

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1. **Introduction.** We use E^n to denote Euclidean n-space, S^n to denote an n-sphere in E^{n+1}, i.e., the set of all points in E^{n+1} at a distance of 1 from the origin. I^{n+1} is the closure of the bounded complementary domain of S^n in E^{n+1}.

A set U in E^n is simply connected if each closed curve J in U can be shrunk to a point in U. That is, if g is a continuous function from $\text{Bd}(I^2)$ onto J, then g may be extended to take all of I^2 into U. Each complementary domain of S^n in E^{n+1} is simply connected.

Alexander [1] described a simple surface, M (a set homeomorphic to S^2), in S^3 such that one complementary domain, U, of M in S^3 was not simply connected. Bing in [3] described a solid horned sphere as $M + U$; that is, Alexander's horned sphere, M, together with the nonsimply connected complementary domain, U, of M.

Bing proved in [3] that if two solid horned spheres were sewn together along the boundary of each with the identity homeomorphism, then the resulting continuum is homeomorphic to S^3.

Ball in [2] modified Alexander's example to obtain a horned sphere such that if two of Ball's solid horned spheres were sewn together in a particular way, the resulting continuum is not S^3. This was done as a partial answer to the question raised by Bing in [3], which may be roughly stated as follows: if two solid horned spheres are sewn together with an arbitrary homeomorphism, is the resultant continuum S^3? The theorem in §3 states that this question has an affirmative answer if the horned sphere in question is the Alexander example.

We shall adopt the following notation. Suppose b is a positive integer with n digits, and each digit in b is either a 1 or a 2. Then we let $b = n\alpha$. (At times it will be necessary to distinguish two or more such positive integers. In this case we shall use $n\alpha$, $n\beta$, etc.) It follows that 1α is 1 or 2 and 2α is either 11, 12, 21, or 22. Suppose some $n\alpha$ is given and we wish to express the positive integer such that the first n digits are exactly the

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Presented to the Society, April 12, 1962; received by the editors December 30, 1963.

[1] This paper is the author's Doctoral Dissertation written at the University of Wisconsin under the supervision of Professor R. H. Bing whose interest, patience and numerous helpful suggestions were of invaluable assistance in its preparation. This research was supported under grant NSF G-11665 while the author was an assistant to R. H. Bing during the summers of 1960 and 1961.

same as $n\alpha$ and the last digit is a 1. We call this integer $n\alpha 1$. If the last digit were a 2, then it would be represented by $n\alpha 2$. For example let $n\alpha = 12112$, then $n\alpha 1 = 121121$ and $n\alpha 2 = 121122$. The positive integer n contained in $n\alpha$ is important in the argument.

Further we shall assume that $\sum A_{n\alpha}$ means the union of all A_n where $A_{n\alpha}$ denotes a set and $n\alpha$ runs through the set of all positive integers containing n digits and each digit is a 1 or a 2. In the same way $\{A_{n\alpha}\}$ is the collection of all $A_{n\alpha}$ where $n\alpha$ takes on all values.

While this notation is unfamiliar, it avoids double subscripts and some of the confusion resulting from notation.

Further if A is a set, $\mathrm{Cl}(A)$ is the closure of A, $\mathrm{Int}(A)$ is the interior of A, and $\mathrm{Bd}(A)$ is the boundary of A. We let d be the Euclidean metric for E^3. For t real, let $a \leqq t \leqq b$ be $[a,b]$, $a < t < b$ be (a,b), $a \leqq t < b$ be $[a,b)$, and $a < t \leqq b$ be $(a,b]$.

2. **The horned sphere.** The following definition of a solid horned sphere is due to Bing in [3]. The notation used in describing the horned sphere will be used in describing a homeomorphism of the sum of two solid horned spheres onto S^3.

Suppose C is a right circular cylinder in E^3 with bases D_1 and D_2. Two mutually exclusive discs in $D_{1\alpha}$ are replaced by the surfaces of tubes $T_{1\alpha 1}$ and $T_{1\alpha 2}$ and discs $D_{1\alpha 1}$ and $D_{1\alpha 2}$ as shown in Figure 1 where $D_{1\alpha 1}$ and $D_{1\alpha 2}$ are the bases of a right circular cylinder $C_{1\alpha}$ and $D_1 + T_{11} + C_1 + T_{12}$ is hooked to $D_2 + T_{21} + C_2 + T_{22}$ as shown.

Discs in the bases of the cylinder $C_{1\alpha}$ are replaced by the surfaces of tubes $T_{1\alpha 11}$, $T_{1\alpha 12}$, $T_{1\alpha 21}$, and $T_{1\alpha 22}$ and discs $D_{1\alpha 11}$, $D_{1\alpha 12}$, and $D_{1\alpha 22}$ as before. The process is continued to get the horned sphere M. We use M_1 to denote the part of M which is the closure of the part of M on the exterior of $C_1 + C_2$, $M_1 = C - \sum(\text{discs cut from } D_1 + D_2) + \sum T_{2\alpha}$. Likewise, M_n denotes the closure of the part of M on the exterior of $\sum C_{n\alpha}$. It is topologically equivalent to S^2 minus 2^{n+1} open discs. Let M_0 be the Cantor set $M - \sum_{n=1}^{\infty} M_n$.

Although M is homeomorphic to S^2, its interior U is not simply connected [4]. We call this complementary domain of M which is not simply connected the *bad complementary domain* of M. A horned sphere plus its bad complementary domain is called a *solid horned sphere*. The part of U which is on the exterior of $\sum C_{n\alpha}$ is denoted by U_n.

3. **The sum of two solid horned spheres.** We now state our main result in the following theorem, which shows the effect of sewing two solid horned spheres together.

THEOREM. *A continuum is homeomorphic with S^3 if it is the sum of three mutually exclusive sets M, U^1, and U^2 such that there is a homeomorphism*

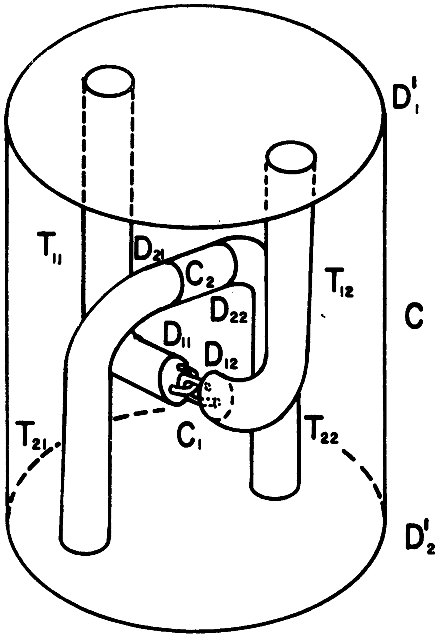

FIGURE 1

of $M + U^i$, $i = 1, 2$, *onto a solid horned sphere that carries M onto a horned sphere.*

We defer the proof of the Theorem until §5.

4. **A decomposition of** E^3. Suppose g is any homeomorphism of M onto M. There is a continuum $(M + U^1 + U^2)_g$ and homeomorphisms g_1 and g_2 such that g_i takes $M + U^i$, $i = 1, 2$, onto a solid horned sphere and such that $(g_2|M)(g_1|M)^{-1} = g$. Now suppose h is a homeomorphism from M onto M and $(M + U^1 + U^2)_h$ is defined as above. Further suppose that there is a topological 2-cell W containing $M_0 + g(M_0)$ contained in M such that $g|W = h|W$. It is easy to see that $(M + U^1 + U^2)_g$ is homeomorphic to $(M + U^1 + U^2)_h$.

As $M_0 + g(M_0)$ is a Cantor set in M, a topological 2-sphere, there is a point y contained in $M - (M_0 + g(M_0))$ such that $g(y)$ is contained in $M - (M_0 + g(M_0))$. There is a topological 2-cell W contained in M containing $M_0 + g(M_0)$ such that the points y and $g(y)$ are contained in $M - W$. As $\mathrm{Cl}(M - W)$ is a topological 2-cell, there is a homeomorphism h of M onto M such that $h(y) = y$ and $g|W = h|W$. It follows that we may assume without loss of generality that g has a fixed point, y, in $M - M_0$.

We now slightly modify the procedure of Bing in [3]. Suppose P is the

xy plane in E^3, i.e., $P = \{x,y,z \in E^3 | z = 0\}$. Suppose $\mathrm{Cl}(A_1)$ and $\mathrm{Cl}(A_2)$ are two solid tori contained in E^3 as used by Bing, such that each is symmetric with respect to P and the boundary, $\mathrm{Cl}(A_i) - A_i$, of A_i is a 2-dimensional torus which intersects P in two circles. We use L to denote that part of E^3 where z is positive, i.e., $\{x,y,z \in E^3 | z > 0\}$, and R to denote that part of E^3 where z is negative, i.e., $\{x,y,z \in E^3 | z < 0\}$.

We shall now describe a decomposition of E^3. As we describe that part of the decomposition that intersects $\mathrm{Cl}(L)$ we shall describe a homeomorphism from $M + U - (M_0 + y)$ into $\mathrm{Cl}(L)$. The notation is the same as used in §2.

There is a homeomorphism F_1 of $\mathrm{Cl}(U_1)$ onto $\mathrm{Cl}(L) - (A_1 + A_2)$ such that $F_1(M_1 - y) = P - (A_1 + A_2)$ and $F_1(\mathrm{Cl}(U) \cdot C_{1\alpha}) = \mathrm{Cl}(L) \cdot (\mathrm{Cl}(A_{1\alpha}) - A_{1\alpha})$. Let $LA_{1\alpha} = \mathrm{Cl}(A_{1\alpha}) \cdot \mathrm{Cl}(L)$, as shown in Figure 2.

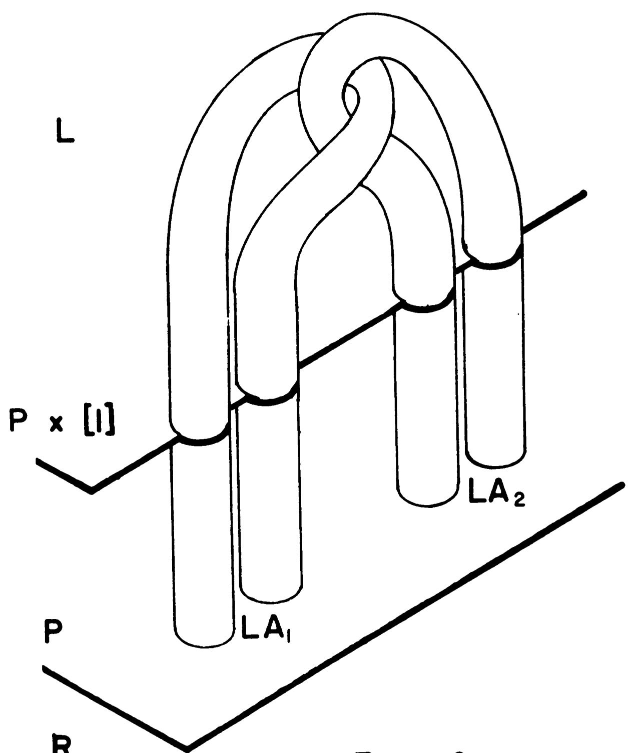

FIGURE 2

Consider $A_{1\alpha}$. Following Bing's procedure [3], put a chain of two solid tori, $\mathrm{Cl}(A_{1\alpha 1}) + \mathrm{Cl}(A_{1\alpha 2})$, in $A_{1\alpha}$ so that each is symmetric with P. There is a homeomorphism F_2 of $\mathrm{Cl}(U_2)$ onto $\mathrm{Cl}(L) - \sum A_{2\alpha}$ that preserves F_1 on $\mathrm{Cl}(U_2)$ such that $F_2(\mathrm{Cl}(U_2) \cdot C_{2\alpha}) = \mathrm{Cl}(L) \cdot (\mathrm{Cl}(A_{2\alpha}) - A_{2\alpha})$. Let $LA_{2\alpha} = \mathrm{Cl}(L) \cdot \mathrm{Cl}(A_{2\alpha})$, as shown in Figure 3.

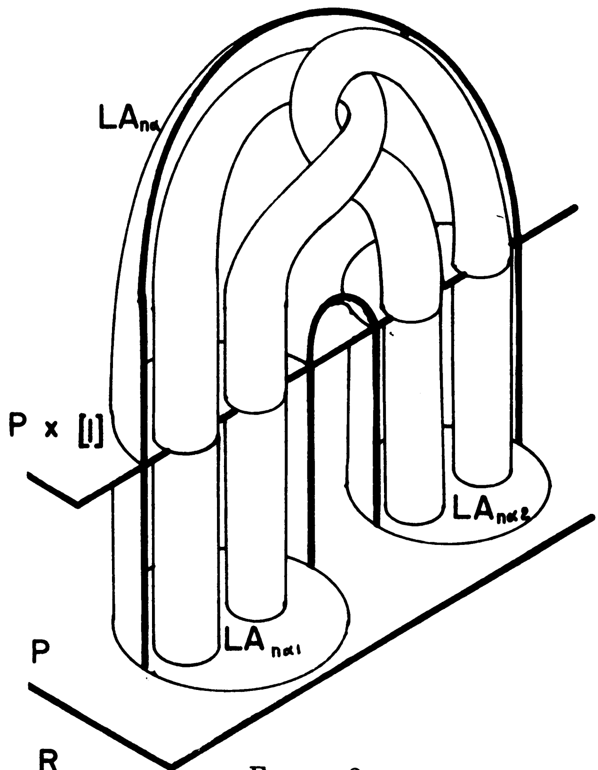

FIGURE 3

The process is continued. We get more solid tori, more F_i's, and more $LA_{i\alpha}$'s. We use LA_0 to denote $\prod_{i=1}^{\infty}\sum LA_{i\alpha}$.

The tori could have been defined so that each component of LA_0 is an arc which intersects no plane parallel to P in two points and so that part of $LA_{i\alpha}$ between P and a plane parallel to P one unit from P in L is the union of two perpendicular solid cylinders, as shown in Figures 2 and 3. We therefore assume this was done and that each component of LA_0 is an arc intersecting P in an end point.

The sequence $F_1, F_2, \cdots$ describes a homeomorphism F between $M+U-(M_0+y)$ and $\mathrm{Cl}(L)-LA_0$. There is a homeomorphism f of $M-y$ onto P such that $f|M-M_0=F|M-M_0$.

Let ϕ be a homeomorphism of $\mathrm{Cl}(L)$ onto $\mathrm{Cl}(R)$ such that $\phi(x,y,z) = (fgf^{-1}(x,y), -z)$, for $z \geqq 0$, where (x,y) denotes the point of E^3, $(x,y,0)$. The homeomorphism ϕ is well defined as fgf^{-1} is a homeomorphism of P onto itself. The homeomorphism ϕ maps any ray in $\mathrm{Cl}(L)$ perpendicular to P and with end point in P onto a ray in $\mathrm{Cl}(R)$ perpendicular to P and with end point in P. Let $RA_{i\alpha}=\phi(LA_{i\alpha})$, and $RA_0=\prod_{i=1}^{\infty}\sum R_{i\alpha}=\phi(LA_0)$. Let $G=\phi F$. The homeomorphism G describes a homeomorphism from $M+U-(M_0+y)$ into $\mathrm{Cl}(R)$.

Now consider $(M + U^1 + U^2)_g$ and the functions g_1 and g_2 defined in the first paragraph of this section. We now define a homeomorphism S as follows. Let $Fg_1 | U^1 = S | U^1$ and $Gg_2 | U^2 = S | U^2$. As F, G, g_1, and g_2 are homeomorphisms and U^1 and U^2 are disjoint open sets, S is a homeomorphism of $U^1 + U^2$ onto $L + R - (LA_0 + RA_0)$. Let $\{u_i\}$ be any sequence of points in $U^1 + U^2$ converging to a point $u \neq y$ contained in $(M + U^1 + U^2)_g$. Let $S(u) = \operatorname{Lim} S(u_i)$. It follows that S is a homeomorphism from $(M + U^1 + U^2)_g - y$ onto the decomposition space X whose points are the points of $E^3 - (LA_0 + RA_0)$ and the components of $LA_0 + RA_0$.

5. **The shrinking of the components of** $LA_0 + RA_0$. If S could have been been defined so that $S(M_0 + g(M_0))$ is a Cantor set of points rather than a Cantor set of arcs, then S would be a homeomorphism from $(M + U^1 + U^2)_g - y$ onto E^3 and with the one point compactification of E^3 the proof of the Theorem would be complete. One way to prove the Theorem is to define a continuous function, T, from E^3 onto E^3 such that T shrinks each distinct component of $LA_0 + RA_0$ onto a distinct point of E^3 and is a homeomorphism of $E^3 - (LA_0 + RA_0)$. It follows that S composed with T is a homeomorphism of $(M + U^1 + U^2)_g - y$ onto E^3. We shall not define the function T directly but define a sequence of homeomorphisms, T_i, that will eventually shrink the components of $LA_0 + RA_0$.

The following lemma will allow us to define the T_i's. However before stating the lemma let us make the following definition. Suppose ϵ is a positive number. Let $\sum LA_{n\alpha}(\epsilon) = (\sum LA_{n\alpha}) \cdot P \times [-\epsilon, 0]$ where $[-\epsilon, 0]$ is an interval on the z axis of E^3. In the same way let $\sum RA_{n\alpha}(\epsilon) = (\sum RA_{n\alpha}) \cdot P \times [0, \epsilon]$. The set $\sum LA_{n\alpha}(\epsilon)$ is a small extension of $\sum LA_{n\alpha}$ into R.

LEMMA 1. *For each $\epsilon > 0$ and each pair of positive integers p and q there is a pair of positive integers r and s and a homeomorphism T' of E^3 onto E^3 fixed on $E^3 - (\sum LA_{p\alpha} + \sum RA_{q\alpha} + \sum LA_{p\alpha}(\epsilon) + \sum RA_{q\alpha}(\epsilon))$ and taking each component of $\sum LA_{r\alpha} + \sum RA_{s\alpha}$ into a set of diameter less than ϵ.*

Proof. The proof of this lemma will be simplified if we make the following assumptions. We will show that these assumptions are valid in §6.

A(1) For each pair of positive integers p and q there is a pair of positive integers p^* and q^*, $p \leqq p^*$, $q \leqq q^*$, such that:

(i) each component of $(\sum LA_{p^*\alpha} + \sum RA_{q^*\alpha}) \cdot P$ is of diameter $\epsilon/5$.

(ii) each component of $\sum LA_{(p^*+1)\alpha} + \sum RA_{(q^*+1)\alpha}$ contains at most one component of $\sum LA_{(p^*+1)\alpha}$.

A(2) For each pair of positive integers p and q there is a pair of positive integers p' and q', $p' > p + 1$, $q' > q + 1$, and a finite set of disks, B_1, $B_2, \cdots, B_k$ contained in P such that,

(i) $(\sum LA_{p'\alpha} + \sum RA_{q'\alpha}) \cdot P$ is contained in $\operatorname{Int}(\sum_{i=1}^{k} B_i)$,

(ii) at most one component of $\sum LA_{p'\alpha}$ intersects B_i, $i = 1, 2, \cdots, k$,

(iii) no component of $\sum LA_{p'\alpha}$ or $\sum RA_{q'\alpha}$ intersects more than one of the disks $B_1, B_2, \cdots, B_k$,

(iv) if B_i intersects both $\sum LA_{p'\alpha}$ and $\sum RA_{q'\alpha}$, then B_i is contained in $\sum \mathrm{Int}((RA_{(q+2)\alpha}) \cdot P)$,

(v) if B_i intersects both $\sum LA_{p'\alpha}$ and $\sum RA_{q'\alpha}$, then B_i is contained in $\sum \mathrm{Int}((LA_{(p+1)\alpha}) \cdot P)$.

A(3)([2]) For each $\epsilon > 0$ and each pair of positive integers p and q there is a homeomorphism, H_0, of E^3 onto E^3 and a finite sequence of planes, $P_1, P_2, \cdots, P_m$, such that:

(i) the distance from P to P_i is less than 1, $i = 1, 2, \cdots, m$,

(ii) P_i is parallel to P, $i = 1, 2, \cdots, m$,

(iii) P_1 and P_2 are in L,

(iv) $P_{(m-1)}$ and P_m are in R,

(v) P_i is between $P_{(i-1)}$ and $P_{(i+1)}$, $i = 2, 3, \cdots, (m-1)$,

(vi) H_0 is fixed on $E^3 - (\sum LA_{p\alpha} + \sum RA_{q\alpha})$,

(vii) the diameter of each component of

$$H_0\left(\sum LA_{(p+1)\alpha} + \sum RA_{(q+1)\alpha} - \sum_{i=1}^{m} P_i\right)$$

is less than $\epsilon/5$.

A(4) Suppose p and p' are the positive integers described in A(2), P_1 and P_2 are the planes described in A(3), and $LA_{p'\alpha}$ is contained in $LA_{(p+1)\alpha}$. There is a homeomorphism H_1, of E^3 onto E^3 such that:

(i) H_1 is fixed on that part of $LA_{(p+1)\alpha}$ that is between P_2 and P,

(ii) H_1 is fixed on that part of $LA_{p'\alpha}$ that is between P_1 and P,

(iii) H_1 is fixed on $E^3 - LA_{(p+1)\alpha}$,

(iv) H_1 takes each $LA_{p'\beta}$ $\alpha \neq \beta$, contained in $LA_{(p+1)\alpha}$ into a set that does not intersect P_1.

A(5) Suppose q is the positive integer of A(2), P_m and $P_{(m-1)}$ are the planes of A(3), and $RA_{(q+1)\alpha}$ is given. Then there is a homeomorphism H_2 of E^3 onto E^3 such that:

(i) H_2 is fixed on that part of $RA_{(q+1)\alpha}$ between $P_{(m-1)}$ and P,

(ii) H_2 is fixed on that part of $RA_{(q+1)\alpha 1}$ between P_m and P,

(iii) H_2 is fixed on $E^3 - RA_{(q+1)\alpha}$,

(iv) $H_2(RA_{(q+1)\alpha 2})$ does not intersect P_m.

Suppose that there is a homeomorphism T' of E^3 onto E^3 fixed on $E^3 - (\sum LA_{p^*\alpha} + \sum RA_{q^*\alpha} + \sum LA_{p^*\alpha}(\epsilon) + \sum RA_{q^*\alpha}(\epsilon))$. If $p^* \geqq p$ and $q^* \geqq q$, then T' is fixed on $E^3 - (\sum LA_{p\alpha} + \sum RA_{q\alpha} + \sum LA_{p\alpha}(\epsilon) + \sum RA_{q\alpha}(\epsilon))$.

([2]) We note that the procedure at this point differs from Bing's construction in [3] in the following way. Bing used the planes $P_1 \cdots P_m$ to partition his sets into "small" pieces whereas the author used $H_0(\sum_{i=1}^{m} P_i)$ to partition his sets. It will be noted that there are some large components of $(\sum LA_{p\alpha} + \sum RA_{q\alpha}) - \sum_{i=1}^{m} P_i$ but there are no large components of $H_0((\sum LA_{(p+1)\alpha} + \sum RA_{(q+1)\alpha}) - \sum_{i=1}^{m} P_i)$.

It follows that we may assume without loss of generality that $p = p^*$ and $q = q^*$ where p^* and q^* are the positive integers promised by A(1).

Now suppose that there is a homeomorphism H of E^3 onto E^3 fixed on $E^3 - (\sum LA_{(p+1)\alpha} + \sum RA_{(q+1)\alpha} + \sum LA_{(p+1)\alpha}(\epsilon) + \sum RA_{(q+1)\alpha}(\epsilon))$ taking each component of $\sum LA_{r\alpha} + \sum RA_{s\alpha}$ into a set that intersects at most five components of $(\sum LA_{(p+1)\alpha} + \sum RA_{(q+1)\alpha}) - \sum_{i=1}^{m} P_i$ where $P_1, \cdots, P_m$ are the planes of A(3). By the properties of H_0 given in A(3), the homeomorphism H_0H is the required homeomorphism T'.

We shall now construct the homeomorphism H inductively using the number of planes in A(3). Let $m = 4$, P_1 and P_2 be in L and P_3 and P_4 be in R. Let p', q' and $B_1, \cdots, B_k$ be the positive integers and disks promised by A(2). Let $B_1, B_2, \cdots, B_n$ be the disks in $B_1, \cdots, B_k$ that intersect $\sum LA_{p'\alpha}$ and $\sum RA_{q'\alpha}$. By A(2) (iv) and A(2) (v), $\sum_{i=1}^{n} B_i$ is a subset of $\sum LA_{(p+1)\alpha} + \sum RA_{(q+1)\alpha}$. Let $K_1, K_2, \cdots, K_u$ be the components of $\sum LA_{(p+1)\alpha} + \sum RA_{(q+1)\alpha}$ and $Q_{1,1}, Q_{1,2}, \cdots, Q_{2,1}, \cdots, Q_{u,v}$ be the components of $\sum LA_{p'\alpha} + \sum RA_{q'\alpha} + \sum_{i=1}^{n} B_i$ where $Q_{i,j}$ is a subset of K_i.

Note that the set $(\sum LA_{p'\alpha} + \sum RA_{q'\alpha}) \cdot P$ may be a very complicated set. We use the disks $B_1, B_2, \cdots, B_n$ to expand $(\sum LA_{p'\alpha} + \sum RA_{q'\alpha}) \cdot P$ into a more manageable set.

If a component $LA_{p'\alpha}$ of $\sum LA_{p'\alpha}$ does not intersect a B_i, $i \leqq n$, then $LA_{p'\alpha}$ does not intersect $\sum RA_{q'\alpha}$. There is a natural method to shrink $LA_{p'\alpha}$ leaving $E^3 - (\sum LA_{p\alpha} + \sum LA_{p\alpha}(\epsilon))$ fixed. In the same way if $RA_{q'\alpha}$ does not intersect a B_i, $i \leqq n$, then $RA_{q'}$ may be shrunk. With this note we ignore sets of this type as they give no difficulty in the shrinking process.

So we may assume $Q_{1,1}$ contains a component of $\sum LA_{p'\alpha}$, say $LA_{p'\alpha}$. There is a homeomorphism H_1 as defined in A(4). By A(1) (ii), K_1 contains exactly one component of $\sum LA_{(p+1)\alpha}$. Hence $H_1(Q_{1,1})$ intersects P_1 and $H_1(Q_{1,j})$, $j \neq 1$, does not intersect P_1.

By A(2) (iii), $Q_{1,1}$ contains exactly one disk B_i, $i \leqq n$. By A(2) (iv), B_i is contained in $\mathrm{Int}((RA_{(q+2)\alpha}) \cdot P)$, where $RA_{(q+2)\alpha}$ is a component of $\sum RA_{(q+2)\alpha}$. Let $RA_{(q+2)\alpha} = RA_{(q+1)\alpha 2}$. By construction $(Q_{1,1}) \cdot R$ is a subset of $RA_{(q+1)\alpha 2}$. There is a homeomorphism H_2 as defined in A(5). Note H_1 is fixed on R and H_2 is fixed on L. Further $H_2(R_{(q+1)\alpha 2})$ does not intersect P_4 and $(Q_{1,1}) \cdot P$ is contained in $RA_{(q+1)\alpha 2}$. It follows that for each integer j, $H_2H_1(Q_{1,j})$ intersects at most three of the four planes P_1, P_2, P_3, and P_4. Further by A(4) and A(5), H_2H_1 is a homeomorphism of E^3 onto E^3 fixed on $E^3 - (\sum LA_{(p+1)\alpha} + \sum RA_{(q+1)\alpha})$.

Without loss of generality we may assume that H_1 and H_2 have been defined for each K_i, $i = 1, 2, \cdots, u$, and that $H_2H_1(Q_{i,j})$ intersects at most three of the four planes P_1, P_2, P_3, P_4 for each i, j. In this case let $H = H_2H_1$.

To complete the proof of this case we must show that no component of $H(\sum LA_{p'\alpha} + \sum RA_{q'\alpha})$ intersects more than five components of

$(\sum LA_{(p+1)\alpha} + \sum RA_{(q+1)\alpha}) - \sum_{i=1}^{4} P_i$. By A(2) (iv) we know that $\sum_{i=1}^{n} B_i$ is contained in $\sum \text{Int}((RA_{(q+2)\alpha}) \cdot P)$ and hence each component $Q_{i,j}$ intersects at most one component of $(\sum RA_{(p+1)\alpha}) \cdot P$. By A(2) (v) $\sum_{i=1}^{n} B_i$ is contained in $\text{Int}((LA_{(p+1)\alpha}) \cdot P)$. Hence each component $Q_{i,j}$ intersects at most one component of $(\sum LA_{(p+1)\alpha}) \cdot P$.

Conditions A(2) (iv) and A(2) (v) together imply that $B_i \times [-1,1]$, $i \leqq n$, is contained in $\text{Int}(LA_{(p+1)\alpha} + RA_{(q+1)\alpha})$. By A(4) and A(5), H is fixed between the planes P_2 and P_3. As H is fixed between the planes P_2 and P_3, no component of $H(\sum LA_{p'\alpha} + \sum RA_{q'\alpha})$ can intersect more than one component of $\sum LA_{(p+1)\alpha} - (P_1 + P_2)$ that intersects P or more than one component of $\sum RA_{(q+1)\alpha} - (P_3 + P_4)$ that intersects P. Further for each $H(Q_{i,j})$ there is a pair of components $LA_{(p+1)\alpha}$ and $RA_{(q+1)\alpha}$ such that $H(Q_{i,j})$ is contained in $LA_{(p+1)\alpha} + RA_{(q+1)\alpha}$. As $H(Q_{i,j})$ can intersect either at most P_1, P_2, and P_3 or at most P_2, P_3, and P_4 it follows that $H(Q_{i,j})$ can intersect at most five components of $(LA_{(p+1)\alpha} + RA_{(q+1)\alpha}) - \sum_{i=1}^{4} P_i$ as is shown in Figure 4, and the lemma follows from this case.

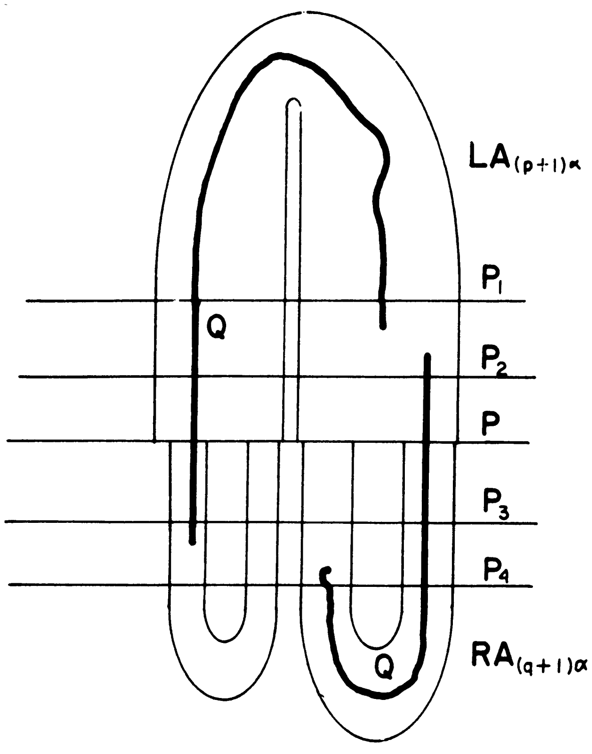

FIGURE 4

We proceed to the case where the number of planes is five, P_1 and P_2 are contained in L, P_4 and P_5 are contained in R and $P_3 \neq P$. Let us proceed as above where there are four planes by first choosing integers p' and q', constructing the B_i's and defining a homeomorphism H_1 and a homeomorphism H_2 using P_4 in place of P_3 and P_5 in place of P_4.

By A(2) for the pair of positive integers p' and q' there is a pair of positive integers p'' and q'' and a finite set of disks $B_1', B_2' \cdots B_{k'}'$ with the properties given in A(2) with appropriate changes in notation. Let Q be any component of $H_2H_1(\sum LA_{p'\alpha} + \sum RA_{q'\alpha})$ that intersects four of the five planes $P_1, \cdots, P_5$. As Q is connected either Q intersects P_1, P_2, P_3, and P_4 or Q intersects P_2, P_3, P_4, and P_5.

Suppose Q intersects P_1, P_2, P_3, and P_4 and P_3 is between P and P_4. By A(4) (i) and A(4) (ii) there are exactly two components R_1 and R_2 of $Q - (P_1 + P)$ whose closures intersect both P_1 and P. By construction $\mathrm{Cl}(R_i)$, $i = 1, 2$, is a right circular cylinder. In §6 we shall show that under this condition we may define a homeomorphism H_1' similar to H_1 and with the properties of H_1 given in A(4) with appropriate changes in notation.

Further by A(5) (i) each component of $Q - (P_4 + P)$ whose closure intersects both P and P_4 is a right cylinder. As P_3 is between P and P_4, P_3 cuts each component of $Q - (P_4 - P)$ in a disk. In §6 we shall show that under these conditions we may define a homeomorphism, H_2', similar to H_2 and with the properties of H_2 given in A(5) with appropriate changes in notation. It follows that each component of

$$H_2'H_1'((H_1H_2(\sum LA_{p''\alpha} + \sum RA_{q''\alpha})) \cdot Q)$$

intersects at most three of the five planes $P_1, \cdots, P_5$.

Now suppose Q intersects P_2, P_3, P_4, and P_5 and P_3 is between P_2 and P. We note that the closure of the two components of $Q - (P_2 + P)$ whose closures intersect both P_2 and P are right circular cylinders, and P_3 cuts each of these components in a disk. Further we note that the closure of each component of $Q - (P + P_3)$ whose closures intersect both P and P_5 is a right cylinder. It follows by the remarks above that we may define a homeomorphism, H_1' and a homeomorphism H_2' such that

$$H_2'H_1'((H_2H_1(\sum LA_{p''\alpha} + \sum RA_{q''\alpha})) \cdot Q)$$

intersects at most three of the five planes $P_1, \ldots, P_5$.

Now P_3 can not be both between P and P_2 and between P and P_4. But actually all of P which we use here is $P \cdot Q$. The obvious procedure is to move $P \cdot Q$ between the "right" two planes and keep the geometry which will allow us to define H_1' and H_2'.

Suppose $Q \cdot (H_2H_1(LA_{p'\alpha})) \neq \emptyset$ and $Q \cdot (H_2H_1(RA_{q'\alpha})) \neq \emptyset$. (The procedure in the contrary case is obvious.) Then $P \cdot Q$ is contained in B_i', $i \leqq n$. By construction $B_i' \times [t_1, t_5]$ is contained in $\mathrm{Int}(\sum LA_{(p+1)\alpha} + \sum RA_{(q+1)\alpha})$ where t_1 is the distance from P to P_1 and $-t_5$ is the distance from P to P_5.

As $P \cdot Q$ is contained in $\operatorname{Int} B_i'$ there is a polyhedral disk Y_i contained in $\operatorname{Int}(B_i')$ such that $(B_i' - Y_i) \cdot (\sum LA_{p'\alpha} + \sum RA_{q'\alpha}) = \emptyset$. Let γ be the least integer such that Q intersects P_γ and η be the greatest integer such that Q intersects P_η. (In this case γ is either 1 or 2 and η is either 4 or 5.) Let t_j be a real number such that $P \times [t_j]$ is a plane between $P_{\gamma+1}$ and $P_{\eta-1}$. There is a piece-wise linear homeomorphism, H_3, from E^3 onto E^3 such that:

(i) H_3 is fixed on $E^3 - (B_i' \times [t_\gamma, t_\eta])$ where t_γ is the distance of P_γ from P and $-t_\eta$ is the distance from P_η to P,

(ii) if $H_3(x) \neq x$, then the line through $H_3(x)$ and x is perpendicular to P,

(iii) $H_3(Y_i \times [t])$, t a real number, is contained in a plane parallel to P,

(iv) $H_3(Y_i) = Y_i \times [t_j]$.

Suppose $H_2H_1(LA_{p'\alpha})$ is contained in Q. Then $(H_3H_2H_1(LA_{p'\alpha}))$ $(B_i \times [t_\gamma, t_\eta]) = H_3((LA_{p'\alpha}) \cdot P) \times [t_\gamma, t_j]$. If $H_2H_1(RA_{q'\alpha})$ is contained in Q then $(H_3H_2H_1(RA_{q'\alpha})) \cdot (B_i \times [t_\gamma, t_\eta]) = H_3((RA_{q'\alpha}) \cdot P \times [t_j, t_\eta]$. Hence as $H_3(P \cdot Q)$ is between the "right" planes and the geometry of $H_3(Q)$ is essentially the same as Q, it follows that a homeomorphism H_1' with the properties of H_1 and a homeomorphism H_2' with the properties of H_2 may be defined on $H_3(Q)$.

Further it follows that there is a homeomorphism, $H_3H_2H_1$, of E^3 onto itself fixed on

$$E^3 - (\sum LA_{(p+1)\alpha} + \sum RA_{(q+1)\alpha} + \sum LA_{(p+1)\alpha}(\epsilon) + \sum RA_{(q+1)\alpha}(\epsilon))$$

taking each component of $\sum LA_{p'\alpha} + \sum RA_{q'\alpha}$ into a set that intersects at most four of the five planes $P_1, \cdots, P_5$. There is a homeomorphism, $H_2'H_1'$, fixed on $E^3 - (H_3H_2H_1(\sum LA_{p'\alpha} + \sum RA_{q'\alpha}))$ together with an $\epsilon/2$ neighborhood of $H_3((\sum LA_{p'\alpha}) \cdot P + (\sum RA_{q'\alpha}) \cdot P)$ taking each component of $H_3H_2H_1(\sum LA_{p''\alpha} + \sum RA_{q''\alpha})$ into a set intersecting at most three of the five planes $P_1, \cdots, P_5$. Let $H = H_2'H_1'H_3H_2H_1$. It follows that each component of $H_0H(\sum LA_{p''\alpha} + \sum RA_{q''\alpha})$ is of diameter less than ϵ.

The general case follows exactly in the same way as the case where $k = 5$. First we find p' and q' and define $H_3H_2H_1$ so that each component of $H_3H_2H_1(\sum LA_{p'\alpha} + \sum RA_{q'\alpha})$ intersects at most $m - 1$ of the m planes $P_1, \cdots, P_m$. Then we find p'' and q'' and define $H_3'H_2'H_1'$ such that each component of $H_3'H_2'H_1'H_3H_2H_1(\sum LA_{p''\alpha} + \sum RA_{q''\alpha})$ intersects at most $m - 2$ of the planes $P_1, \cdots, P_m$. We then continue finding more p's and q's and defining H_3's and H_2's and H_1's until $H(\sum LA_{r\alpha} + \sum RA_{s\alpha})$ intersects at most three of the m planes $P_1, \cdots, P_m$, where H is the composition of the H_3's, H_2's, and H_1's in proper order and r is the last p chosen and s is the last q chosen. We find that we have proven the lemma modulo A(1), A(2), A(3), A(4), and A(5).

Lemma 1 is a restatement of Bing's Lemma in [3]. Following Bing we now prove the theorem.

Proof of theorem. We find from Lemma 1 that there is a pair of positive integers $r(1)$ and $s(1)$ and a homeomorphism T_1 of E^3 onto E^3 that leaves each point of $E^3 - (\sum LA_{1\alpha} + \sum RA_{1\alpha} + \sum LA_{1\alpha}(1/2) + \sum RA_{1\alpha}(1/2))$ fixed and takes each component of $\sum LA_{r(1)\alpha} + \sum RA_{s(1)\alpha}$ into a set of diameter less than $1/2$.

There is a positive integer ϵ such that for each set A of diameter less than ϵ, $T_1(A)$ is of diameter less than $1/2^2$. By Lemma 1 there is a pair of positive integers $r(2)$ and $s(2)$ and a homeomorphism T_2' of E^3 onto E^3 that leaves each point of

$$E^3 - (\sum LA_{r(1)\alpha} + \sum RA_{s(1)\alpha} + \sum LA_{r(1)\alpha}(\epsilon) + \sum RA_{s(1)\alpha}(\epsilon))$$

fixed and takes each component of $\sum LA_{r(2)\alpha} + \sum RA_{s(2)\alpha}$ into a set of diameter less than ϵ. Let $T_2 = T_1 T_2'$. T_2 is a homeomorphism of E^3 onto itself that takes each component of $\sum LA_{r(2)\alpha} + \sum RA_{s(2)\alpha}$ into a set of diameter less than $1/2^2$ and $T_2 = T_1$ on

$$E^3 - (\sum LA_{r(1)\alpha} + \sum RA_{s(1)\alpha} + \sum LA_{r(1)\alpha}(1/2) + \sum RA_{s(1)\alpha}(1/2)).$$

Continuing the process we obtain a sequence of homeomorphisms $T_1, \cdots, T_n, \cdots$ such that $T_{n+1} = T_n$ on

$$E^3 - (\sum LA_{r(n)\alpha} + \sum RA_{s(n)\alpha} + \sum LA_{r(n)\alpha}(1/2^n) + \sum RA_{s(n)\alpha}(1/2^n)),$$

and the diameter of each component of $T_{n+1}(\sum LA_{r(n+1)\alpha} + \sum RA_{s(n+1)\alpha})$ is less than $1/2^{n+1}$.

It follows that $T = \text{Lim}\, T_1, T_2, \cdots, T_n, \cdots$ is a continuous function of E^3 onto E^3 such that the image of each distinct component of $LA_0 + RA_0$ is a distinct point of E^3. Hence T is a homeomorphism taking the decomposition space X onto E^3. As S is the function, defined at the end of §4, from $(M + U^1 + U^2)_g - y$ onto X, it follows that TS is a homeomorphism from $(M + U^1 + U^2)_g - y$ onto E^3 and the theorem follows with the one point compactification of E^3.

6. **The details of Lemma 1.** By construction $\prod_{n=1}^{\infty}(\sum LA_{n\alpha}) \cdot P$ is a Cantor set of points in P. Hence for each $\epsilon > 0$ there is a positive integer n such that the diameter of each component of $(\sum LA_{n\alpha}) \cdot P$ is of diameter less than ϵ.

We use this to find the pair of positive integers p^* and q^*, given the pair of positive integers p and q and an $\epsilon > 0$, as required in A(1). There is a positive integer $p^* > p$ such that the diameter of each component of $(\sum LA_{p^*\alpha}) \cdot P$ is of diameter less that $\epsilon/15$. Let δ be the minimum of $\epsilon/15$ and the distances between pairs of components of $(\sum LA_{p^*\alpha}) \cdot P$. There is a positive integer $q^* > q + 1$ such that the diameter of each component of $(\sum RA_{(q^*-1)\alpha}) \cdot P$ is less than δ. It follows that each component of $(\sum LA_{p^*\alpha}) \cdot P + (\sum RA_{q^*\alpha}) \cdot P$ is of diameter less that $\epsilon/5$. For let $RA_{q^*\alpha}$ be a component of $\sum RA_{q^*\alpha}$. Now $(RA_{q^*\alpha}) \cdot P$ is contained on the interior of some component of $(\sum RA_{(q^*-1)\alpha}) \cdot P)$. As δ was chosen so small that no component of $(\sum RA_{(q^*-1)\alpha}) \cdot P$ intersects more than one component of $(\sum LA_{p^*\alpha}) \cdot P$, $RA_{q^*\alpha}$ can intersect at most one component of $\sum LA_{p^*\alpha}$. Therefore each component of $\sum LA_{p^*\alpha} + \sum RA_{q^*\alpha}$ can contain at most one component of $\sum LA_{p^*\alpha}$. It follows that we may choose p^* and q^* with the desired properties given in A(1).

The positive integers p' and q' of A(2) are chosen in approximately the same way. Let δ_1 be the minimum of the distance from $(\sum RA_{(q+3)\alpha}) \cdot P$ to $P - (\sum RA_{(q+2)\alpha}) \cdot P$ and the distances between pairs of components of $(\sum RA_{(q+3)\alpha}) \cdot P$. There is a positive integer $r > p$ such that the diameter of each component of $(\sum LA_{r\alpha}) \cdot P$ is less than δ_1. Note that if a component, K, of $(\sum LA_{r\alpha}) \cdot P$ intersects $\sum RA_{(q+3)\alpha}$ then K is contained in $\mathrm{Int}((\sum RA_{(q+2)\alpha}) \cdot P)$.

Let δ_2 be the minimum of the distance from $(\sum LA_{(r+1)\alpha}) \cdot P$ to $P - (\sum LA_{r\alpha}) \cdot P$ and the distance between pairs of components of $(\sum LA_{(r+1)\alpha}) \cdot P$. There is a positive integer s such that each component of $(\sum RA_{s\alpha}) \cdot P$ is of diameter less than δ_2.

We shall now choose the B_i's. Let the components of $(\sum LA_{r\alpha}) \cdot P$ be called $V_1, V_2, \cdots, V_n$. We note that each V_k contains two components, $U_{k,1}$ and $U_{k,2}$, of $(\sum LA_{(r+1)\alpha}) \cdot P$. Further δ_2 was chosen so small that no component of $(\sum RA_{s\alpha}) \cdot P$ can intersect more than one of the three sets $P - V_k$, $U_{k,1}$ and $U_{k,2}$, $k = 1, 2, \cdots, n$. Hence for each $U_{k,h}$ there is a polyhedral simple closed curve $J_{k,h}$ in $(\sum LA_{r\alpha}) \cdot P - ((\sum LA_{(r+1)\alpha}) \cdot P + (\sum RA_{s\alpha}) \cdot P)$ such that the $J_{k,h}$'s are pair-wise disjoint, and $J_{k,h}$ separates $U_{k,h}$ from $(\sum LA_{(r+1)\alpha}) \cdot P - U_{k,h}$. Let $B_1, \cdots, B_{2n}$ be the polyhedral disks contained in P whose boundaries are $J_{1,1}, J_{1,2}, J_{2,1}, \cdots, J_{n,2}$. Let B_{2n+1} be a disk in $P - \sum_{i=1}^{2n} B_i$ such that if Z is a component of $(\sum RA_{s\alpha}) \cdot P$ that does not intersect a B_i then Z is contained in $\mathrm{Int}(B_{2n+1})$.

We note that each B_i contains at most one $U_{h,k}$. As each component of $\sum LA_{(r+2)\alpha}$ intersects P in the interior of some $U_{h,k}$, each B_i intersects at most one component of $LA_{(r+2)\alpha}$ for $i \leqq 2n$. Further each component of $\sum LA_{(r+2)\alpha}$ can intersect at most one B_i. We let $p' = r + 2$ and $q' = s + 1$, and we find that:

(i) $(\sum LA_{p'\alpha} + \sum RA_{q'\alpha}) \cdot P$ is contained in $\mathrm{Int}(\sum_{i=1}^{k} B_i)$,

(ii) at most one component of $\sum LA_{p'\alpha}$ intersects B_i, $i = 1, 2, \cdots, (2n+1)$,

(iii) no component of $\sum LA_{p'\alpha}$ or of $\sum RA_{q'\alpha}$ intersects more than one of the disks $B_1, \cdots, B_{(2n+1)}$.

Suppose B_i intersects $\sum LA_{p'\alpha}$ and B_i intersects $\sum RA_{q'\alpha}$. Let $U_{k,h}$ be the component of $(\sum LA_{(r+1)\alpha}) \cdot P$ contained by B_i. It follows that B_i is contained in V_k, a component of $(\sum LA_{r\alpha}) \cdot P$ that intersects $RA_{(q+3)\alpha}$. Therefore:

(iv) if B_i intersects both $\sum LA_{p'\alpha}$ and $\sum RA_{q'\alpha}$, then B_i is contained in $\sum \mathrm{Int}((RA_{(q+2)\alpha}) \cdot P)$,

(v) if B_i intersects both $\sum LA_{p'\alpha}$ and $\sum RA_{q'\alpha}$, then B_i is contained in $\sum \text{Int}((LA_{(p+1)\alpha}) \cdot P)$.

Hence the positive integers p' and q' and the disks $B_1, B_2, \cdots, B_k$ may be found with the properties given in A(2).

Before we construct H_0, H_1, and H_2, we note that each $LA_{n\alpha}$ is the homeomorphic image of the corresponding $C_{n\alpha}$. As $C_{n\alpha}$ is a solid right circular cylinder, there is a natural homeomorphism from $I^2 \times [0,1]$ onto $C_{n\alpha}$. This implies that there is a natural homeomorphism from $I^2 \times [0,1]$ onto $LA_{n\alpha}$ where $(I^2 \times [0] + (I^2 \times [1])$ is taken into $(LA_{n\alpha}) \cdot P$. We may now define a new metric, ρ, on $LA_{n\alpha}$ induced by the natural homeomorphism. That is, if ψ is the natural homeomorphism and x and y are two points in $LA_{n\alpha}$, then $\rho(x,y) = d(\psi^{-1}(x), \psi^{-1}(y))$ where d is the Euclidean metric for $I^2 \times [0,1]$. Therefore, it is meaningful to speak of a piecewise linear homeomorphism of $LA_{n\alpha}$ onto itself with respect to the induced metric ρ. In fact the following lemma may be applied to any $LA_{n\alpha}$ using the induced metric ρ.

LEMMA 2. *Suppose A is a closed subset of $I^2 \times [0,1]$ such that* $\text{Bd}(I^2) \times [0,1]$ *does not intersect A, and t_1, t_2, t_3, t_4, s_1, and s_2 are six real numbers, $0 \leqq t_1 < t_2 < t_3 < t_4 \leqq 1$, $t_1 < s_1 < s_2 < t_4$. Then there is a piece-wise linear homeomorphism of $I^2 \times [0,1]$ onto itself fixed on* $\text{Bd}(I^2) \times [0,1] + I^2 \times [0,t_1] + I^2 \times [t_4, 1]$ *such that:*

(i) *the image of (x,t) is (x,s) where $x \in I^2$ and, t and $s \in [0,1]$,*

(ii) *the image of $(I^2 \times [t]) \cdot A$ is a subset of $I^2 \times [s], 0 \leqq t \leqq 1$, and $0 \leqq s \leqq 1$,*

(iii) *the image of $(I^2 \times [t_2]) \cdot A$ is contained in $I^2 \times [s_1]$ and the image of $(I^2 \times [t_3]) \cdot A$ is contained in $I^2 \times [s_2]$.*

Indication of proof. There is a disk D contained on the interior of I^2 such that A is contained in $D \times [0,1]$. At this point the proof follows by writing out the proper linear homeomorphism on the sets $I^2 \times [0,t_1]$, $I^2 \times [t_4, 1]$, $D \times [t_1, t_2]$, $D \times [t_2, t_3]$, $D \times [t_3, t_4]$, $(I^2 - \text{Int}(D)) \times [t_1, t_2]$, $(I^2 - \text{Int}(D)) \times [t_2, t_3]$, and $(I^2 - \text{Int}(D)) \times [t_3, t_4]$.

The homeomorphism promised by Lemma 2 will be called a *Basic Homeomorphism*. If ϕ is a Basic Homeomorphism of a 3-cell, C^*, contained in E^3, then ϕ is a homeomorphism of C^* onto itself and ϕ is fixed on $\text{Bd}(C^*)$. Hence ϕ can be extended to a homeomorphism of E^3 onto itself where $\phi | (E^3 - \text{Int}(C^*))$ is the identity.

We may now define H_0 as promised in A(3). By A(1) we may assume that the diameter of each component of $(\sum LA_{p\alpha} + \sum RA_{q\alpha}) \cdot P$ is less than $\epsilon/5$. Choose any plane P' in L parallel to P and at a distance of less than 1 from P. Choose any component of $\sum LA_{p\alpha}$, say $LA_{p\alpha}$. Let ϕ be a homeomorphism from $I^2 \times [0,1]$ onto $LA_{p\alpha}$ such that $\phi((I^2 \times [0]) + (I^2 \times [1])) = (LA_{p\alpha}) \cdot P$ and $\phi((I^2 \times [t_2]) + (I^2 \times [t_3])) = (LA_{p\alpha}) \cdot P'$. As LA_0 is a Cantor set of arcs and each component of $(LA_{p\alpha}) \cdot P$ is of diameter less than $\epsilon/5$, we may assume that the diameter of $\phi(I^2 \times [t])$ is less than

$\epsilon/5$ for $0 \leqq t \leqq 1$. Let $t_0 = 0$ and $t_4 = 1$. We have now defined the positive integers t_1, t_2, t_3, and t_4 of Lemma 2. As the diameter of $\phi(I^2 \times [t])$ is less than $\epsilon/5$ for $0 \leqq t \leqq 1$ there is a pair of positive numbers s_1 and s_2, $0 < t_2 < s_1 < s_2 < t_3 < 1$, such that the diameter of $\phi(I^2 \times [s_1, s_2])$ is less than $\epsilon/5$. Let the closed set of Lemma 2 be $LA_{p\alpha 1} + LA_{p\alpha 2}$. Hence we apply Lemma 2 and the construction of H_0 is obvious when we note that the Basic Homeomorphism is uniformly continuous.

We shall now define H_1 as promised in A(4). Suppose $LA_{(p+1)\alpha}$ is any component of $\sum LA_{(p+1)\alpha}$ and $LA_{p'\alpha}$ is a component of $(\sum LA_{p'\alpha}) \cdot (LA_{(p+1)\alpha})$. Let P_1 and P_2 be defined as in A(3). Let R_1 and R_2 be the closure of the two components of $LA_{(p+1)\alpha} - P_1$ that intersects both P_1 and P. As R_i, $i = 1, 2$, is a right cylinder, there is a homeomorphism ϕ_1 of $I^2 \times [0, 1]$ such that:

(i) $\phi_1(I^2 \times [0, t_1] = R_1$,

(ii) $\phi_1(I^2 \times [t_3, 1]) = R_2$,

(iii) $\phi_1(I^2 \times [t])$, $0 < t < t_1$, $t_3 < t < 1$, is a disk contained in a plane parallel to P which irreducibly separates $LA_{p\alpha}$.

It follows that $P_2 \cdot (LA_{(p+1)\alpha}) = \phi_1(I^2 \times [t_5]) + \phi_1(I^2 \times [t_4])$ for some t_5 and t_4, $0 < t_5 < t_1 < t_3 < t_4 < 1$. We may assume without loss of generality that $LA_{p'\alpha} = LA_{(p+1)\alpha,1,1,\cdots,1,1}$. Let $R_1 \cdot (LA_{(p+1)\alpha 1}) \neq \emptyset$. There is a pair of positive numbers s_1 and s_2 such that $t_3 < s_1 < s_2 < t_4$. By construction $(\phi_1(I^2 \times [t_1])) \cdot (LA_{(p+1)\alpha 2}) \neq \emptyset$. Hence there is a positive number t_2, $t_1 < t_2 < t_3$ such that $(\phi_1(I^2 \times [t_2])) \cdot (LA_{(p+1)\alpha 2}) = \emptyset$. We have chosen the six positive numbers t_1, t_2, t_3, t_4, s_1, and s_2. Hence there is a Basic Homeomorphism ψ_1 as defined in Lemma 2 such that:

(i) ψ_1 is fixed on the closure of the two components of $LA_{(p+1)\alpha 1} - P_1$ that intersect both P_1 and P,

(ii) ψ_1 is fixed on that part of $LA_{(p+1)\alpha}$ between P_2 and P,

(iii) ψ_1 is fixed on $E^3 - LA_{(p+1)\alpha}$,

(iv) $\psi_1(LA_{(p+1)\alpha 2})$ does not intersect P_1.

If $LA_{p'\alpha} = LA_{(p+1)\alpha 1}$ let $\psi_1 = H_1$.

Suppose $LA_{p'\alpha} \neq LA_{(p+1)\alpha 1}$. By (i) in the above paragraph we note that the closure of the two components of $\psi_1(LA_{(p+1)\alpha 1}) - P_1$ that intersect both P and P_1 are right cylinders. We also note that the only condition for the definition of ψ_1 was that R_1 and R_2 be right cylinders. It follows that a homeomorphism ψ_2 with properties given for ψ_1 in the above paragraph may be defined by substituting $\psi_1(LA_{(p+1)\alpha 1})$ for $LA_{(p+1)\alpha}$. If $LA_{p'\alpha} = LA_{(p+1)\alpha 1,1}$ let $H_1 = \psi_2\psi_1$. If $LA_{p'\alpha} \neq LA_{(p+1)\alpha 1,1}$, then there is a homeomorphism ψ_3. It follows that in a finite number of steps the homeomorphism H_1 will be defined with the properties of A(4). Further as the homeomorphism H_3H_1 takes the closure of the two components of $LA_{n\alpha} - P_1$ that intersects both P and P_1 into right cylinders the homeomorphism H_1' may be defined.

The homeomorphism H_2 given in A(5) is defined in exactly the same way as ψ_1 where $LA_{(p+1)\alpha}$ is replaced by $RA_{(q+1)\alpha}$, P_1 is replaced by P_n, and P_2 is replaced by $P_{(n-1)}$.

We have supplied the details of Lemma 1 and have completed the proof of the theorem.

Bibliography

1. J. W. Alexander, *An example of a simply connected surface bounding a region which is not simply connected,* Proc. Nat. Acad. Sci. U.S.A. **10** (1924), 8-10.

2. B. J. Ball, *The sum of two solid horned spheres,* Ann. of Math. (2) **69** (1959), 253-257.

3. R. H. Bing, *A homeomorphism between the* 3*-sphere and the sum of two solid horned spheres,* Ann. of Math. (2) **56** (1952), 354-362.

4. W. A. Blankinship and R. H. Fox, *Remarks on certain pathological open subsets of* 3*-space and their fundamental group,* Proc. Amer. Math. Soc. **1** (1950), 618-624.

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