ON THE CATEGORY OF INDECOMPOSABLE DISTRIBUTIONS ON TOPOLOGICAL GROUPS

BY

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1. **Introduction.** According to a theorem of A. I. Khinchine [1] any distribution on the real line can be written as the convolution of two distributions one of which is the convolution of a finite or a countable number of indecomposable distributions and the other is an infinitely divisible distribution without indecomposable factors. Further, any distribution which is not infinitely divisible has an indecomposable component. This result gives an indication of the existence of a large collection of indecomposable distributions. It is, however, not clear from this result alone that there exists a nonatomic or absolutely continuous indecomposable distribution. This question was raised by H. Cramér [2] and an answer in the affirmative was given by P. Lévy [3]. However, what is available is only a meagre supply of examples [3; 4; 5] even in the case of the real line. In this connection there arises naturally the question of the "size" of the class $\mathfrak{M}_1$, of indecomposable distributions. Or more precisely, what is the category of $\mathfrak{M}_1$? The object of this paper is to answer questions of this type.

These questions are studied here under the general framework of a complete separable metric group G. We consider three classes of distributions: (1) all indecomposable distributions, (2) all nonatomic indecomposable distributions and (3) all indecomposable distributions which are absolutely continuous with respect to the Haar measure in the case of a locally compact abelian group. It is shown that, under appropriate conditions on G, the indecomposable distributions form a dense G_δ in the class of all distributions in cases (1) and (2) and in the class of all absolutely continuous distributions in case (3). This lends substance to the statement that, in general, a distribution is indecomposable. It may be added that most of these results seem to be new even in the case when G is the real line.

In this connection it is interesting to compare these results with that of W. Rudin [6]. Rudin considers the convolution algebra $L_1(G)$ of all functions integrable with respect to the Haar measure in a locally Euclidean group G and shows that every element in $L_1(G)$ is decomposable. The decomposability of all elements is made possible by the fact that the factors are not confined to non-negative elements of $L_1(G)$ as we have done in our work.

Our analysis incidentally throws some light on the problem of existence of nonatomic measures in a separable metric space. It follows as a consequence of our work that the set of all nonatomic measures in a complete

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separable metric space without isolated point is a dense G_δ in the set of all measures under the weak topology.

2. **Preliminaries.** Throughout the paper we suppose that G denotes a complete separable metric group. Additional assumptions on G will be specifically mentioned as and when necessary. We employ the customary notation of denoting the group operation as xy, $x, y \in G$ in the case of general groups and as $x+y$ if G is abelian. e always denotes the unit in G. For any two subsets A, B of G we write $AB=[z: z=xy,\ x\in A,\ y\in B]$ and $A^{-1} = [z: z^{-1}\in A]$ (in case the group is abelian we use instead the symbols $A+B$ and $-A$ respectively).

The convolution operation. By a measure (or distribution) we mean a probability measure defined on the σ-field $\mathfrak{B}$ of Borel subsets of G. Let $\mathfrak{M}$ denote the collection of all probability measures on $\mathfrak{B}$. For $\mu, \nu \in \mathfrak{M}$ the convolution $\mu * \nu$ is defined as follows

$$(\mu*\nu)(A) = \int \mu(Ax^{-1})d\nu(x). \tag{2.1}$$

With this operation $\mathfrak{M}$ becomes a semigroup which is abelian if and only if G is so. It should be noted that $\mu * \nu$ in (2.1) can be written in the equivalent form

$$(\mu*\nu)(A) = \int \nu(x^{-1}A)d\mu(x).$$

For each $g\in G$ and $\mu \in \mathfrak{M}$, $\mu * g$ denotes the right translate of μ by g i.e., the measure $\mu(Eg^{-1})$. $g * \mu$ is defined similarly. By a translate of μ we mean a measure of the form $\mu * g$ or $g * \mu$. Regarding the measure-theoretic terminology used but unaccompanied by explanation, we follow Halmos [7].

DEFINITION 2.1. A measure λ is *decomposable* if and only if there exist two nondegenerate measures μ and ν such that $\lambda = \mu * \nu$. In the contrary case λ is said to be *indecomposable.*

DEFINITION 2.2. A nondegenerate measure α is said to be a factor of a measure β if and only if there exists a measure γ such that either $\beta = \alpha * \gamma$ or $\beta = \gamma * \alpha$.

We shall denote by $\mathfrak{M}_0$ the set of all decomposable measures and $\mathfrak{M}_1$ the set of all indecomposable measures.

DEFINITION 2.3. The spectrum of a measure μ is the smallest closed set $A \subset G$ such that $\mu(A)=1$.

The existence of the spectrum is well known and it is also easy to see that if A, B, and C are the spectra of the measures μ, ν and $\mu * \nu$ respectively then C = closure of (AB).

Topologies in $\mathfrak{M}$. In the sequel we shall be mainly concerned with the weak topology in $\mathfrak{M}$. It is defined through convergence as follows:

Definition 2.4. A sequence of measures $\{\mu_n\}$ converges weakly to a measure μ if and only if, for every real valued bounded continuous function f defined on G, $\int f d\mu_n \to \int f d\mu$.

It is clear that the class of subsets of $\mathfrak{M}$, of the form

$$\left[\mu: \left| \int f_i d\mu - \int f_i d\mu_0 \right| < \epsilon_i, i = 1, 2, \cdots, k\right]$$

where $(f_1, \cdots, f_k)$ is any finite set of bounded continuous functions and $(\epsilon_1, \cdots, \epsilon_k)$ is any finite set of positive numbers forms a neighbourhood system for the weak topology in $\mathfrak{M}$. It is useful to note that the sets of the type

$$[\mu: \mu(V_i) > \mu_0(V_i) - \epsilon_i, i = 1, 2, \cdots, k]$$

where $\epsilon_i > 0$ for all i and V_i are open subsets of G, are open in the weak topology.

Now we shall gather a few results about the weak topology in $\mathfrak{M}$ which we need in the sequel.

Theorem 2.1 (Prohorov [8], Varadarajan [9]). *If G is a complete separable metric space, the space $\mathfrak{M}$ of measures on G becomes a complete separable metric space under the weak topology.*

Theorem 2.2 (Prohorov [8]). *If G is a complete separable metric space, a subset $M \subset \mathfrak{M}$ is conditionally compact in the weak topology if and only if, for every $\epsilon > 0$, there exists a compact set $K_\epsilon \subset G$ such that $\mu(K_\epsilon) > 1 - \epsilon$ for every $\mu \in M$.*

Theorem 2.3 (Ranga Rao [10]). *In a complete separable metric space G a sequence $\mu_n \in \mathfrak{M}$ converges weakly to $\mu \in \mathfrak{M}$ if and only if the following holds: For every class $\mathfrak{A}$ of continuous functions on G such that*

(i) *$\mathfrak{A}$ is uniformly bounded,*

(ii) *$\mathfrak{A}$ is compact in the topology of uniform convergence on compacta,*

$$\lim_{n \to \infty} \sup_{f \in \mathfrak{A}} \left| \int f d\mu_n - \int f d\mu \right| = 0.$$

All topological notions in $\mathfrak{M}$ used in §§1–6 refer to the weak topology. Only in the last section, we find it necessary to consider the strong topology induced by the norm $\|\mu\| = \sup_{A \in \mathfrak{B}} |\mu(A)|$, $\mu \in \mathfrak{M}$.

Indecomposable distributions on the real line. We now state two results due to P. Lévy [3] concerning absolutely continuous indecomposable distributions on the real line. We shall have occasion to use the latter one in the last section.

Let Z be any real-valued random variable which takes values in a bounded interval. We denote by $[Z]$ the integral part of Z and the conditional distribution of Z given that $[Z] = n$, by μ_n.

THEOREM 2.4. *Let Z be any real-valued random variable taking values in a bounded interval and satisfying the following properties:*

(a) *$[Z]$ is even with probability one,*

(b) *the distribution of $[Z]$ is indecomposable,*

(c) *the family of distributions μ_n, n running over possible values of $[Z]$, has no common factor.*

Then the distribution of Z is indecomposable.

THEOREM 2.5. *Let μ_1, μ_2 denote the uniform distributions on the intervals $[a, b]$ and $[c, d]$ respectively. If $(b-a)/(d-c)$ is irrational then μ_1 and μ_2 have no common factor.*

3. **Shift compactness in $\mathfrak{M}$.** In the theory of sums of independent random variables we often come across the situation where a sequence of distributions fails to converge to any limit but actually does converge when suitably centered (see Gnedenko and Kolmogorov [11]). In this section we shall make a systematic analysis of this phenomenon in relation to the convolution operation between distributions on groups. To this end, it is convenient to introduce the following

DEFINTION 3.1. A family $\mathfrak{N}$ is said to be shift compact if, for every sequence $\mu_n \in \mathfrak{N}$ $(n=1, 2, \cdots)$, there is a sequence of measures ν_n such that (1) ν_n is a translate of μ_n and (2) ν_n has a convergent subsequence.

The main result of this section is the following theorem which reveals an important structural property of the topological semigroup in relation to the notion of shift compactness. Its applications in the theory of factorisation of distributions in groups will be discussed in a separate paper.

THEOREM 3.1. *Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ be three sequences of measures on G such that*

$$\lambda_n = \mu_n * \nu_n \qquad (n = 1, 2, \cdots). \tag{3.1}$$

If the sequence $\{\lambda_n\}$ is conditionally compact, then each one of the sequences $\{\mu_n\}$ and $\{\nu_n\}$ is shift compact.

As an immediate consequence we have the following

COROLLARY 3.1. *For any $\lambda \in \mathfrak{M}$, the family $\mathfrak{F}(\lambda)$ of all factors of λ is shift compact.*

Before proceeding to the proof of Theorem 3.1 we shall establish the following

LEMMA 3.1. *Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ be three sequences of measures on G such that $\lambda_n = \mu_n * \nu_n$ for each n. If the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are conditionally compact then so is the sequence $\{\nu_n\}$.*

Proof. Since the sequences λ_n and μ_n are conditionally compact it follows from Theorem 2.2 that, given $\epsilon>0$, there exists a compact set K_ϵ such that

$$\lambda_n(K_\epsilon) > 1 - \epsilon, \qquad \mu_n(K_\epsilon) > 1 - \epsilon$$

for all n. Then we have

$$(3.2) \qquad \begin{aligned} 1 - \epsilon < \lambda_n(K_\epsilon) &= \int \nu_n(x^{-1}K_\epsilon)d\mu_n(x) \\ &\leqq \int_{K_\epsilon} \nu_n(x^{-1}K_\epsilon)d\mu_n(x) + \epsilon \end{aligned}$$

or

$$(3.3) \qquad \int_{K_\epsilon} \nu_n(x^{-1}K_\epsilon)d\mu_n(x) > 1 - 2\epsilon.$$

(3.3) implies the existence of a point $x_n \in K_\epsilon$ with the property

$$\nu_n(x_n^{-1}K_\epsilon) > 1 - 3\epsilon$$

and consequently we have

$$\nu_n(K_\epsilon^{-1}K_\epsilon) > 1 - 3\epsilon$$

for all n. Since $K_\epsilon^{-1}K_\epsilon$ is compact and independent of n, another application of Theorem 2.2 leads to the fact that the sequence $\{\nu_n\}$ is conditionally compact. This completes the proof of the lemma.

Proof of Theorem 3.1. We choose a sequence ϵ_r of positive numbers such that $\sum \epsilon_r < \infty$. Then the conditional compactness of the sequence $\{\lambda_n\}$ implies, by Theorem 2.2, that there exists a sequence of compact sets K_r such that

$$\lambda_n(K_r) > 1 - \epsilon_r, \qquad r = 1, 2, \cdots$$

for all n. Now choose a positive sequence η_r descending to zero and satisfying

$$\sum_{r=1}^{\infty} \epsilon_r \eta_r^{-1} \leqq \frac{1}{2} \cdot$$

Let

$$(3.4) \qquad E_{nr} = [x: \mu_n(K_r x^{-1}) > 1 - \eta_r], \qquad F_n = \bigcap_{r=1}^{\infty} E_{nr}.$$

Then, from (3.1) and (3.4), we have

$$\begin{aligned} 1 - \epsilon_r \leqq \lambda_n(K_r) &= \int_{E_{nr}} \mu_n(K_r x^{-1})d\nu_n(x) + \int_{E'_{nr}} \mu_n(K_r x^{-1})d\nu_n(x) \\ &\leqq \nu_n(E_{nr}) + (1 - \eta_r)\nu_n(E'_{nr}) \end{aligned}$$

where E'_{nr} denotes the complement of the set E_{nr}. Thus we obtain

$$\nu_n(E'_{nr}) \leqq \frac{\epsilon_r}{\eta_r}$$

and consequently

$$\nu_n(F_n') \leqq \sum_n \epsilon_r \eta_r^{-1} \leqq \frac{1}{2} \cdot$$

Hence $F_n \neq \phi$. Let x_n be any element in F_n. Then, from the definition of F_n, we have

$$\mu_n(K_r x_n^{-1}) > 1 - \eta_r \tag{3.5}$$

for all n and all r.

We now write $\alpha_n = \mu_n * x_n$ (the right translate of μ_n by x_n) and $\beta_n = x_n^{-1} * \nu_n$. Then, obviously, $\lambda_n = \alpha_n * \beta_n$ and from (3.5) and Theorem 2.2 it follows that both the sequences λ_n and α_n are conditionally compact. Lemma 3.1 now implies that β_n is conditionally compact. The fact that α_n and β_n are translates of μ_n and ν_n respectively completes the proof of the theorem.

4. The class $\mathfrak{M}_1$ is a G_δ. The purpose of this section is simply to prove the following.

THEOREM 4.1. *Let G be a complete separable metric group. Then the class $\mathfrak{M}_1$ of all indecomposable distributions forms a G_δ.*

Before proceeding to the proof of this theorem we make a digression in order to pick up a few auxiliary facts.

Let $f_1, f_2, \cdots$, be a sequence of bounded functions on G with the following properties

(a) for each j, $f_j(x)$ is uniformly continuous in both the right and left uniformities of G.

(b) the sequence $\{f_j\}$ separates points of G.

The existence of such a sequence of functions may be seen as follows. Since G is a separable metric group there is a sequence of neighbourhoods $\{N_i\}$ of e such that $\bigcap_{i=1}^{\infty} N_i = \{e\}$. Then by a well known result (cf. A. Weil [16, pp. 13–14]) there exists a sequence of functions $f_i(x)$ $(i=1, 2, \cdots)$ which are uniformly continuous in the two-sided uniformity (i.e., in both the right and left uniformities) and such that $f_i(e)=0$ and $f_i(x)=1$ for $x \notin N_i$. Let $\{x_n\}$ be a sequence dense in G. Then the countable family $S=\{\phi_{ij}\}$ where $\phi_{ij}(x)=f_i(xx_j)$ possesses both the properties. It is only necessary to prove that the family S separates points of G. If not let a and b be two distinct elements of G such that $\phi_{ij}(a)=\phi_{ij}(b)$ for all i and j or equivalently $f_i(ax_j) = f_i(bx_j)$ for all i and j. Since $\{x_j\}$ is dense it follows that $f_i(ab^{-1}) = f_i(e) = 0$ for all i. But $ab^{-1} \notin N_i$ for some i and hence $f_i(ab^{-1})=1$ for some i. This contradiction shows that $S=\{\phi_{ij}\}$ separates points of G.

In all that follows, $S=\{f_j\}$ is a fixed sequence with the above properties.

It is then clear that a measure μ on G is degenerate if and only if the induced measure μf_j^{-1} on the real line is degenerate for each j. For any real valued bounded continuous function f and any measure μ, we write

$$V(f,\mu) = \sup_{a\in G} \max\left\{\left[\int f^2(ax)d\mu - \left(\int f(ax)d\mu\right)^2\right], \left[\int f^2(xa)d\mu - \left(\int f(xa)d\mu\right)^2\right]\right\}. \tag{4.1}$$

It is obvious that a measure μ is degenerate if and only if $V(f_j,\mu)=0$ for all j.

LEMMA 4.1. *If f is bounded and uniformly continuous in both the right and left uniformities of the group G and μ_n is a sequence of measures converging weakly to μ, then*

$$\lim_{n\to\infty} V(f,\mu_n) = V(f,\mu).$$

Proof. For each f which is bounded and uniformly continuous in the right as well as the left uniformity, it is clear that each one of the families of functions $\{f(xa),\ a\in G\}$, $\{f^2(xa),\ a\in G\}$, $\{f(ax),\ a\in G\}$, $\{f^2(ax),\ a\in G\}$ is uniformly bounded and equi-continuous at each point of G. Consequently, they are conditionally compact in the topology of uniform convergence on compacta. The lemma is then an immediate consequence of Theorem 2.3.

Let now $E_{ij}(\epsilon)$ be defined as follows

$$E_{ij}(\epsilon) = [\mu:\mu = \alpha * \beta,\ V(f_i,\alpha) \geqq \epsilon,\ V(f_j,\beta) \geqq \epsilon] \tag{4.2}$$

where f_i and f_j are any two functions from S. Then we have

LEMMA 4.2. *For any $\epsilon>0$ and each i, j the set $E_{ij}(\epsilon)$ is closed.*

Proof. Let μ_n be a sequence of measures in $E_{ij}(\epsilon)$ converging to some measure μ. Then by (4.2) there exist measures α_n and β_n such that

$$\begin{gathered} \mu_n = \alpha_n * \beta_n \\ V(f_i,\alpha_n) \geqq \epsilon, \qquad V(f_j,\beta_n) \geqq \epsilon. \end{gathered} \tag{4.3}$$

From Theorem 3.1, it follows that there exists a sequence $a_n\in G$ such that the sequences of measures $\{\alpha_n * a_n\}$ and $\{a_n^{-1} * \beta_n\}$ are conditionally compact. Thus we can choose subsequences $\alpha_{n_k} * a_{n_k}$ and $a_{n_k}^{-1} * \beta_{n_k}$ converging to some measures α_0 and β_0 respectively. Since μ_n converges to μ and $\mu_{n_k} = \alpha_{n_k} * a_{n_k} * a_{n_k}^{-1} * \beta_{n_k}$ we have $\mu=\alpha_0 * \beta_0$. It is clear from the definition of $V(f,\mu)$ (see (4.1)) that

$$V(f,\alpha_{n_k} * a_{n_k}) = V(f,\alpha_{n_k}). \tag{4.4}$$

From Lemma 4.1 it follows immediately that

$$\lim_{k\to\infty} V(f_i, \alpha_{n_k}) = V(f_i, \alpha_0).$$

Similarly

$$\lim_{k\to\infty} V(f_j, \beta_{n_k}) = V(f_j, \beta_0).$$

Thus from (4.3) we have

$$V(f_i, \alpha_0) \geqq \epsilon, \qquad V(f_j, \beta_0) \geqq \epsilon$$

or $\mu \in E_{ij}(\epsilon)$. This completes the proof.

We shall now prove Theorem 4.1 by showing that the set of all decomposable measures is an F_σ. In fact

$$\mathfrak{M}_0 = \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{r=1}^{\infty} E_{ij}(r^{-1}). \tag{4.5}$$

It is clear that any measure belonging to the right side of (4.5) is decomposable and hence belongs to $\mathfrak{M}_0$. Let now μ be any measure in $\mathfrak{M}_0$. Then there exist two nondegenerate measures α and β such that $\mu = \alpha * \beta$. Since α and β are nondegenerate it follows from the remarks made earlier that there exist two functions f_i and f_j belonging to S, with the property

$$V(f_i, \alpha) > 0, \qquad V(f_j, \beta) > 0.$$

Let $\epsilon = \min[V(f_i, \alpha), \ V(f_j, \beta)]$. Then for $r > 1/\epsilon$, $\mu \in E_{ij}(r^{-1})$. Thus $\mathfrak{M}_0$ is contained in the right side of (4.5). An application of Lemma 4.2 completes the proof of the theorem.

REMARKS. 1. Let $\mathfrak{M}_c$ denote the class of nonatomic measures. It may be noted that the class of all indecomposable nonatomic measures is a G_δ in $\mathfrak{M}_c$ under the relative topology. At the moment, it is not clear whether even a single nonatomic measure exists in G. These points will be clarified in §6.

2. Let G be a locally compact abelian separable metric group and $\mathfrak{A}(G)$ the class of all absolutely continuous distributions in G. Since the norm topology in $\mathfrak{A}(G)$ is stronger than the weak topology it is clear that the set of all indecomposable absolutely continuous distributions is a G_δ in $\mathfrak{A}(G)$ (the relevant topology being the norm topology).

To determine the category of the various classes of indecomposable distributions, it is thus sufficient to find their closures. We note one case where the class $\mathfrak{M}_1$ is of first category. This is the situation when the group G is finite, as is implied in the work of Vorobev [12].

In the rest of the paper we will study the closures of three classes of indecomposable distributions

(1) the general case—the class $\mathfrak{M}_1$ itself;

(2) the nonatomic case—the class of all nonatomic measures in $\mathfrak{M}_1$;

(3) the absolutely continuous case—the class of indecomposable distributions absolutely continuous with respect to the Haar measure in a locally compact abelian group G.

5. **The general case.** Before we state the main theorem of this section we begin with some lemmas, the purpose of which is to construct indecomposable distributions in G. To this end we introduce the following definition.

DEFINITION 5.1. A subset $A \subset G$ is said to be *decomposable* if there exist two sets $A_1, A_2 \subset G$ such that (a) each of A_1, A_2 contains at least two elements and (b) $A_1A_2 = A$; a set $A \subset G$ is said to be *indecomposable* if it is not decomposable.

LEMMA 5.1. *Let A be any countable indecomposable set and μ a measure such that $\mu(A)=1$ and $\mu(\{g\})>0$ for every $g \in A$. Then μ is indecomposable.*

Proof. Let us suppose that μ is decomposable. Then $\mu=\mu_1 * \mu_2$ where μ_1 and μ_2 are nondegenerate measures with mass concentrated at a countable or finite number of points. Let

$$A_i = [g: g \in G, \mu_i(g) > 0], \qquad i = 1, 2.$$

From the conditions of the lemma it follows that $A = A_1A_2$, which contradicts the fact that A is indecomposable.

LEMMA 5.2. *Let B be an infinite countable set $\{g_1, g_2, \cdots\}$ with the following property: $g_rg_s^{-1} \neq g_tg_u^{-1}$ for every set of distinct integers r, s, t, u no two of which are equal. If F is any finite subset of G then the set $B \cup F$ is indecomposable.*

Proof. Suppose the lemma is not true. Then there exist two sets A_1, A_2, at least one of which contains an infinite number of elements and such that $B \cup F = A_1A_2$. Let $A_1 = (x_1, x_2 \cdots x_n \cdots)$ and $y_1, y_2 \in A_2$. Since the elements x_ry_1, $r=1, 2, \cdots$ are all distinct all but a finite number of them belong to B. Thus there exists a finite set N of integers such that

$$x_ry_1 \in B \qquad \text{for } r \notin N.$$

Take any integer $m \notin N$. For at most one integer s, say $s=k_1$, x_sy_1 can be equal to x_my_2. Similarly, for at most one integer, say $s=k_2$, x_sy_2 can be equal to x_my_1. Choose any integer $n \notin N$ and different from k_1 and k_2. Then x_my_1, x_my_2, x_ny_1 and x_ny_2 are all distinct and belong to B. But

$$x_my_1(x_ny_1)^{-1} = x_my_2(x_ny_2)^{-1}$$

which contradicts the defining property of B.

LEMMA 5.3. *If G is an infinite group, then there exists a set B with the property described in Lemma* 5.1.

Proof. Let g_1, g_2, g_3 be any three distinct elements of the group G. Suppose $g_1, g_2, \cdots, g_n$ have been chosen. Consider the set A_n of all elements of the form $g_{i_1}^{\pm 1}g_{i_2}^{\pm 1}g_{i_3}^{\pm 1}$ where i_1, i_2, i_3 are any three positive integers less than or equal to n. Since A_n is finite and the group G is infinite A_n' is nonempty. Choose any element g_{n+1} from A_n'. The sequence $g_1, g_2, \cdots$ chosen in this way has the required property.

THEOREM 5.1. *If the group G is infinite, then $\mathfrak{M}_1$ is a dense G_δ.*

Proof. Any measure in G is a weak limit of measures concentrated at a finite number of points. From Lemmas 5.1–5.3 it is clear that any measure with a finite spectrum is a weak limit of indecomposable distributions. Thus indecomposable distributions are dense in $\mathfrak{M}$. In view of Theorem 4.1, this completes the proof.

6. **The nonatomic case.** To start with we shall investigate the existence of a nonatomic measure in an arbitrary complete separable metric space. This, in itself, seems to be an interesting problem (cf. [13]).

THEOREM 6.1. *Let X be any complete separable metric space without any isolated points. Then there exists a nonatomic measure on X.*

Proof. Let $\mathfrak{M}$ be the class of all probability measures on X. Then by Theorem 2.1 $\mathfrak{M}$ is a complete separable metric space under the weak topology. For any given $\epsilon > 0$, we denote by $C(\epsilon)$ the class of all measures which have at least one atom of mass greater than or equal to ϵ. Then the class of all measures with atomic components can be represented as $\bigcup_{r=1}^{\infty} C(1/r)$. If there does not exist any nonatomic measure, we have

$$\mathfrak{M} = \bigcup_{r=1}^{\infty} C\left(\frac{1}{r}\right).$$

It is not difficult to verify by making use of Theorem 2.2 that $C(\epsilon)$ is closed in the weak topology. Thus by Baire's category theorem, at least one $C(1/r)$ has interior. Hence there exists a measure μ_0 with an atom of positive mass $> \delta > 0$ such that, whenever a sequence of measures μ_n converges weakly to μ_0, μ_n has an atom of mass at least δ for sufficiently large n. Since measures with finite spectrum (i.e., for which spectrum is a finite set) are everywhere dense we can, without loss of generality, assume that μ_0 is a measure with masses $p_1, p_2, \cdots, p_k$ at the points $x_1, x_2, \cdots, x_k$ respectively.

Let $N_n(x_1), N_n(x_2), \cdots, N_n(x_k)$ be sequences of neighbourhoods shrinking to $x_1, x_2, \cdots, x_k$ respectively. We can and do assume that these neighbourhoods are disjoint. Since by assumption X has no isolated points each of these neighbourhoods contains an infinite number of points. We distribute the mass p_i among the points of $N_n(x_i)$ such that the mass at each point is less than $\delta/2$. By doing this for every i and every n we obtain a sequence of measures μ_n converging weakly to μ_0 and such that the mass of μ_n at any point is $\leq \delta/2$. This contradicts the defining property of μ_0 and shows that $C(1/r)$ has no interior for any r. The proof is complete.

COROLLARY 6.1. *Let X be any complete separable metric space with an uncountable number of points. Then there exists a nonatomic measure on X.*

Proof. Let Y denote the set of all accumulation points of X. Then it is well known (cf. Hausdorff [14, p. 146]) that X can be written in the form $X = Y \cup N$ where

(a) Y is closed and dense in itself,

(b) N is countable.

By Theorem 6.1 there exists a nonatomic measure on Y and hence on X.

The proof of Theorem 6.1 actually yields something more. In fact we have

COROLLARY 6.2. *Let X be a complete separable metric space without isolated points. Then the set of all nonatomic measures which give positive mass to each open subset of X is a dense G_δ in $\mathfrak{M}$.*

If G is any nondiscrete complete metric group, then it is clear that it is necessarily uncountable and cannot have any isolated point. Consequently we have the following:

THEOREM 6.2. *Let G be any nondiscrete complete separable metric group (not necessarily abelian). Then the set of all nonatomic indecomposable distributions which give positive mass to each open set is a dense G_δ in $\mathfrak{M}$.*

7. **The absolutely continuous case.** In this section we suppose that G is a locally compact abelian group and consider measures absolutely continuous with respect to the Haar measure on G. Let $\mathfrak{A} = \mathfrak{A}(G)$ denote the collection of these measures. The convergence notion that is appropriate for $\mathfrak{A}$ is the norm convergence of measures or the L_1 convergence of their densities. The main object here is to show that in the sense of this convergence the indecomposable measures in $\mathfrak{A}$ are dense.

In the first instance we develop in the following lemmas a general method of constructing absolutely continuous indecomposable distributions in G.

LEMMA 7.1. *Let A_1, A_2, A_3 be three closed subsets of G satisfying the following conditions:*

$$(1) \qquad (A_i - A_i) \cap (A_j - A_k) = \varnothing \qquad \text{for } i = 1, 2, 3 \text{ and } j \neq k;$$

$$(2) \qquad (A_1 - A_2) \cap (A_2 - A_3) = (A_2 - A_3) \cap (A_3 - A_1) = (A_3 - A_1) \cap (A_1 - A_2) = \varnothing.$$

Let μ_1, μ_2, μ_3 be measures with $\mu_i(A_i) = 1$ and $\lambda = p_1\mu_1 + p_2\mu_2 + p_3\mu_3$ where $p_i > 0$ $(i = 1, 2, 3)$ and $p_1 + p_2 + p_3 = 1$. Then λ is decomposable if and only if μ_1, μ_2, μ_3 have a nondegenerate common factor.

Proof. If μ_1, μ_2, μ_3 have a nondegenerate common factor, it is obvious that λ is decomposable. Conversely let us suppose that λ is decomposable. Then there exist two nondegenerate measures α and β such that $\lambda = \alpha * \beta$. Let G

and D denote the spectra of α and β respectively. It is obvious that

$$C + D \subset A_1 \cup A_2 \cup A_3 = A. \tag{7.1}$$

For each $c \in G$, we write

$$D_i(c) = [d: d \in D \text{ and } c + d \in A_i] = D \cap (A_i - c) \tag{7.2}$$

for $i=1, 2, 3$. The rest of the proof depends on an analysis of the nature of decomposition $\{D_i(c)\}$ of D. It is convenient to divide it into three steps.

I. The sets $D_i(c)$ possess the following properties.

(i) $\bigcup_{i=1}^{3} D_i(c) = D$ for each c.

(ii) $D_i(c) \cap D_j(c) = \emptyset$ for $i \neq j$.

(iii) For any two distinct c_1 and c_2, $D_i(c_1) = D_i(c_2)$ for some i, implies that $D_j(c_1) = D_j(c_2)$ for $j=1, 2, 3$.

(iv) if $c_1 \neq c_2$, $D_i(c_1) \cap D_j(c_2) \neq \emptyset$ implies that $D_i(c_1) = D_j(c_2)$.

The first three properties are very simple. We shall prove (iv). Let us suppose that $D_i(c)$ and $D_j(c)$ have a common point d and $D_i(c_1) \neq D_j(c_2)$. Then there exists a point $d' \in D_i(c_1)$ which is not in $D_j(c_2)$. From (i) above it follows that there is a $k(\neq j)$ such that $d' \in D_k(c_2)$. From these facts we have

$$\begin{aligned} c_1 + d \in A_i, &\qquad c_1 + d' \in A_i, \\ c_2 + d \in A_j, &\qquad c_2 + d' \in A_k. \end{aligned} \tag{7.3}$$

Consequently

$$d - d' \in (A_i - A_i) \cap (A_j - A_k).$$

This contradicts the assumption (1) of the lemma and proves (iv). It should be noted that the property (iv) implies that the decompositions $\{D_i(c)\}$ for $c \in C$ are only permutations of each other.

II. One of the following relations is always satisfied. Either

(a) for each $c \in C$, all but one of $D_i(c)$ are empty, i.e., $D_i(c) = D$ for some i, or

(b) for any two $c_1, c_2 \in C$, $D_i(c_1) = D_i(c_2)$ for $i=1, 2, 3$.

The proof of this is quite straightforward and is similar to that of (iv) above. We shall not go into details.

III. Now we suppose that case (a) obtains. Let $C_i = [c: D_i(c) \neq \emptyset]$. It is then easily verified that (1) C_i's are mutually disjoint and their union is C, (2) $C_i + D \subset A_i$ for each i. Let the measures α_i ($i=1, 2, 3$) be defined as follows:

$$\alpha_i(E) = \alpha(E \cap C_i)/\alpha(C_i).$$

(Note that $\alpha(C_i) > 0$.) It is then not difficult to verify that

$$\alpha_i * \beta = \mu_i \qquad \text{for } i = 1, 2, 3.$$

Thus for each i, β is a factor of μ_i.

In case (b), let $D_i = D_i(c)$. Obviously the D_i's are mutually disjoint and $C + D_i \subset A_i$ for each i. Writing $\beta_i(E) = \beta(E \cap D_i)/\beta(D_i)$ we get as before

$$\alpha * \beta_i = \mu_i \qquad \text{for } i = 1, 2, 3.$$

In this case α is the required common factor. This completes the proof of the lemma.

LEMMA 7.2. *Let G be a noncompact group. Then for any given compact set K, there exist elements $g, h \in G$ such that the sets K, $K+g$, $K+h$ satisfy the conditions* (1) *and* (2) *of Lemma* 7.1.

Proof. It may be verified that conditions (1) and (2) of Lemma 7.1 in this case reduce to choosing g and h such that none of the elements g, h, $g-h$, $g+h$, $2g+h$, $2h-g$ belong to the compact set $C = (K-K)-(K-K)$. Let

$$F = [x : x = 2y, y \in G].$$

Then there are two possibilities.

Case 1. F has compact closure. In this case we can choose an element g such that $g \notin C$ and $\bar{F} \cap (C+g) = \emptyset$. Since G is noncompact such elements exist. Let h be any element such that $h \notin C \cup (C+g) \cup (C-g) \cup (C+2g)$. The pair g, h satisfies our requirements.

Case 2. The closure of F is not compact. Let $g \notin C$ be arbitrary. Since $\bar{F}$ is not compact we can find an $h \in G$ such that $2h \notin C+g$ and $h \notin C \cup (C+g) \cup (C-g) \cup (C-2g) \cup (C+2g)$. As is easily verified the pair g, h serves our purpose. This completes the proof.

LEMMA 7.3. *Let G be an infinite compact metric abelian group. Let A be a subset such that*

$$0 < \lambda(A) < 1, \tag{1}$$

$$\int_A \chi_j(x) d\lambda(x) \neq 0 \qquad \text{for } j = 0, 1, 2, \cdots, \tag{2}$$

where λ is the normalized Haar measure on G and $\chi_0, \chi_1, \cdots$ are the characters of G. If λ_1, λ_2 are defined by

$$\lambda_1(E) = \lambda(E \cap A)/\lambda(A),$$
$$\lambda_2(E) = \lambda(E \cap A')/\lambda(A'),$$

then λ_1 and λ_2 do not have a common factor.

Proof. Let χ_0 be the identity character. Since for the Haar measure $\int \chi_j d\lambda = 0$ for $j \neq 0$ and $0 < \lambda(A) < 1$, we have

$$\int_{A'} \chi_j d\lambda \neq 0 \qquad \text{for every } j. \tag{7.4}$$

We shall now prove that the measures λ_1 and λ_2 cannot have a common factor. If this is not true, then let μ be a common factor. Then there exist measures α_1, α_2 such that

$$\lambda_1 = \alpha_1 * \mu, \qquad \lambda_2 = \alpha_2 * \mu. \tag{7.5}$$

From the definitions of λ_1 and λ_2 and (7.5) we have

$$\lambda(A)\lambda_1 + \lambda(A')\lambda_2 = \lambda = (\lambda(A)\alpha_1 + \lambda(A')\alpha_2) * \mu. \tag{7.6}$$

Taking the characteristic functionals on both sides of (7.5) and (7.6) we get

$$\begin{gathered} \int \chi_j d\lambda_1 = \left(\int \chi_j d\alpha_1\right)\left(\int \chi_j d\mu\right), \\ \int \chi_j d\lambda_2 = \left(\int \chi_j d\alpha_2\right)\left(\int \chi_j d\mu\right), \\ \left(\lambda(A) \int \chi_j d\alpha_1 + \lambda(A') \int \chi_j d\alpha_2\right)\left(\int \chi_j d\mu\right) = 0 \qquad \text{for } j \neq 0. \end{gathered} \tag{7.7}$$

From condition (2) of the lemma and (7.7) we deduce that

$$\int \chi_j d\mu \neq 0 \qquad \text{for all } j.$$

Thus from (7.7) we have

$$\lambda(A) \int \chi_j d\alpha_1 + \lambda(A') \int \chi_j d\alpha_2 = 0 \qquad \text{for all } j \neq 0,$$

which is the same as saying

$$\lambda(A)\alpha_1 + \lambda(A')\alpha_2 = \lambda. \tag{7.8}$$

From (7.5) and the definition of λ_1 and λ_2 we get

$$\int \alpha_1(A' - x)d\mu(x) = \lambda_1(A') = 0,$$

$$\int \alpha_2(A - x)d\mu(x) = \lambda_2(A) = 0.$$

Consequently

$$\alpha_1(A' - x) = 0 \ a\cdot e(\mu),$$
$$\alpha_2(A - x) = 0 \ a\cdot e(\mu).$$

Thus there exists a point x_0 such that

$$\alpha_1(A' - x_0) = \alpha_2(A - x_0) = 0. \tag{7.9}$$

(7.8) and (7.9) imply that

$$\alpha_1(E) = \frac{\lambda(E \cap [A - x_0])}{\lambda(A)} = [\lambda_1 * (-x_0)](E),$$

$$\alpha_2(E) = \frac{\lambda(E \cap [A' - x_0])}{\lambda(A')} = [\lambda_2 * (-x_0)](E).$$

Thus from (7.5) and (7.10) we obtain

$$\lambda_1 = \lambda_1 * (-x_0) * \mu, \qquad \lambda_2 = \lambda_2 * (-x_0) * \mu. \tag{7.11}$$

Taking characteristic functionals on both sides of (7.11) we have

$$\int \chi_j d((-x_0) * \mu) = 1 \qquad \text{for all } j.$$

Thus μ is degenerate at the point x_0. The proof of the lemma is complete.

LEMMA 7.4. *In any infinite compact group G there exists a set A possessing the properties* (1) *and* (2) *of Lemma* 7.3.

Proof. Let $S(\lambda)$ be the measure ring obtained by considering the space of Borel subsets of G modulo λ-null sets. This is a complete metric space with the distance $d(E, F) = \lambda(E \triangle F)$ where E and F belong to $S(\lambda)$ (cf. [7, pp. 165–169]). Let $\chi_0, \chi_1, \cdots$ be the characters of G, χ_0 being the identity. We consider the following mapping from $S(\lambda)$ to the complex plane. For any $E \in S(\lambda)$, we write

$$f_j(E) = \int_E \chi_j d\lambda.$$

The mapping f_j is obviously continuous. Hence the sets

$$V_j = \left[E: \int_E \chi_j d\lambda \neq 0\right]$$

are open in $S(\lambda)$. We shall now prove that each V_j is dense in $S(\lambda)$. Let $A \in S(\lambda)$ and

$$\int_A \chi_j d\lambda = 0.$$

Let $\lambda(A) = c > 0$. Since λ is nonatomic, for any $0 < \epsilon < c$ there exists a set $B \subset A$ such that $\epsilon/2 < \lambda(B) < \epsilon$. Let C be any subset of B for which

$$\int_C \chi_j d\lambda \neq 0.$$

Such a C exists, for otherwise χ_j will vanish in B almost everywhere but at the same time $|\chi_j| = 1$. The set $A \cap C'$ has the property

$$d(A \cap C', A) = \lambda((A \cap C') \triangle A) = \lambda(C) < \epsilon.$$

Since this is true for any sufficiently small ϵ it is possible to get A as a limit of elements belonging to V_j. Since the class of sets A with $\lambda(A) > 0$ is dense in the ring $S(\lambda)$ it follows that the sets V_j are actually dense in $S(\lambda)$. By the Baire category theorem it follows that $\bigcap_{j=1}^{\infty} V_j$ is dense in $S(\lambda)$. Thus there exist Borel sets with the required properties.

LEMMA 7.5. *In any locally compact separable metric abelian group G there exist two absolutely continuous measures with compact supports which do not have a common factor.*

We shall prove this lemma in two steps. First of all let us assume that G is a finite dimensional vector space. Let A_1 and A_2 be two cubes in G such that the ratio of the lengths of their sides is irrational.

Then the uniform distributions μ_1 and μ_2, concentrated in A_1 and A_2 respectively, cannot have a common factor. For, if they have, then at least one of the one-dimensional marginal distributions of μ_1 must have a common factor with the corresponding marginal distribution of μ_2. Since the corresponding marginal distributions of μ_1 and μ_2 are rectangular distributions in the real line with the ratio of the lengths of their supports irrational, it follows from Theorem 2.5 that they cannot have a common factor. This proves the lemma in the case when G is a vector space.

If G is an infinite compact group, we have by Lemma 7.3 two absolutely continuous measures which do not have a common factor.

Now a result of Pontrjagin [15] states that for any general locally compact group G there exists an open subgroup H such that

$$H = V \oplus Z$$

where V is a vector group, Z a compact group and $\oplus$ denotes the direct sum. In V we take any two absolutely continuous measures μ_1 and μ_2 without any common factor. If Z is infinite we take two absolutely continuous measures ν_1 and ν_2 in Z without common factor. If Z is finite we take ν_1 and ν_2 to be any two degenerate measures. We form the product measures

$$\lambda_1 = \mu_1 \times \nu_1, \qquad \lambda_2 = \mu_2 \times \nu_2$$

in H. Since H is open λ_1 and λ_2 are absolutely continuous with respect to the Haar measure in G. Since none of the marginals of λ_1 and λ_2 have a common factor we conclude that λ_1 and λ_2 themselves cannot have a common factor. This completes the proof of the lemma.

THEOREM 7.1. *In any locally compact noncompact complete separable metric* *abelian group G the set of all absolutely continuous indecomposable distributions is a dense G_δ in $\mathcal{A}(G)$.*

That the set under consideration is a G_δ follows from the remarks made in §4. It remains to be proved that it is dense. It is clear that the set of all absolutely continuous measures with compact supports is everywhere dense. Thus it remains only to prove that any absolutely continuous measure μ with compact support is a limit of a sequence of absolutely continuous indecomposable measures. Let the support of μ be K_0. Let μ_1 and μ_2 be two absolutely continuous measures with compact supports K_1 and K_2 and having no common factor. Such measures exist because of Lemma 7.5. Let

$$K = K_0 \cup K_1 \cup K_2.$$

By using Lemma 7.3 we choose two points $g, h \in G$ such that K, $K+g$, $K+h$ satisfy conditions (1) and (2) of Lemma 7.1. We write

$$\alpha_1 = \mu, \qquad \alpha_2 = \mu_1 * g, \qquad \alpha_3 = \mu_2 * h$$

and

$$\mu_n = \left(1 - \frac{2}{n}\right)\alpha_1 + \frac{1}{n}\alpha_2 + \frac{1}{n}\alpha_3, \qquad n \geqq 2.$$

From Lemma 7.1 it follows that μ_n is indecomposable. It is obvious that μ_n converges in norm to α_1 which is the same as μ. This completes the proof of the theorem.

REMARK. In the above theorem the assumption of noncompactness of G has played a crucial role. The question arises—is this assumption necessary? Or, more precisely, if G is an infinite compact group, is the collection of indecomposable distributions in $\mathcal{A}(G)$ dense in $\mathcal{A}(G)$? The answer is not known.

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