INJECTIVE DIMENSION IN NOETHERIAN RINGS

BY

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Introduction. Among Noetherian rings quasi-Frobenius rings are those which are self injective [10, Theorem 18]. This paper is concerned primarily with Noetherian rings whose self injective dimension is finite. Thus, for example, Theorem 3.3, describing rings of self injective dimension one, can be regarded as a one dimensional analogue of the theory of quasi-Frobenius rings. Integral group rings furnish a basic example here. This theorem, noticed independently by Jans [12], was suggested to the author by a problem on torsion free modules to which we apply it in [5], and it is the origin of the present paper.

The balance of the paper consists essentially of elaborations on various aspects of the proof of Theorem 6.3, which characterizes commutative Noetherian rings of finite self injective dimension(2). They are a special class of Macaulay-Cohen rings (see §5 for definition) and enjoy most of the visible properties, locally, of local complete intersections, the latter constituting a fundamental example. For local rings we have, thus, the following hierarchy: regular $\Rightarrow$ local complete intersection $\Rightarrow$ finite self injection dimension $\Rightarrow$ Macaulay-Cohen; and none of the implications can be reversed.

1. **The chain condition.** Injective modules are seldom finitely generated so, when considering them, the conventional uses of chain conditions on the ring are not available. Theorem 1.1 provides formulations of the ascending chain condition which are better adapted for our purposes. The theorem is stronger than necessary for our applications, but it may be of interest for suggesting possibly useful definitions of the chain condition in more general abelian categories.

We begin by recalling the basic facts about injective envelopes. Λ is a ring and all modules are left Λ-modules. A monomorphism $0\rightarrow A\rightarrow E$ of Λ-modules is called *essential* (we also say that A is an *essential submodule* of E) if $A\cap B=0\Rightarrow B=0$ for all submodules B of E. If, in addition, E is injective, we call $0\rightarrow A\rightarrow E$ an *injective envelope* of A, and we sometimes indicate this by writing $E(A)$ for E (the embedding being understood, but undenoted).

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(2) After writing this paper I discovered from Professor Serre that these rings have been encountered by Grothendieck, the latter having christened them "Gorenstein rings." They are described in his setting by the fact that a certain module of differentials is locally free of rank one.

Eckmann and Shopf [9] have shown that injective envelopes always exist. It follows that every Λ-module A has an injective resolution,

$$0 \to A \to E_0 \to \cdots \to E_n \xrightarrow{d} E_{n+1} \to \cdots$$

for which E_{n+1} is an injective envelope of $d(E_n)$; such will be called a *minimal injective resolution.* We write $\mathrm{Id}_\Lambda(A)$ for the injective dimension of a Λ-module A. It is easy to see that, for a minimal resolution of A as above, $\mathrm{Id}_\Lambda(A) < n$ if and only if $E_n = 0$.

THEOREM 1.1. *The following are equivalent for a ring* Λ.

(1) Λ *is left Noetherian.*

(2) *A direct sum of injective left* Λ*-modules is injective.*

(3) *A direct limit of injective left* Λ*-modules is injective.*

(4) *A direct limit of left* Λ*-modules of injective dimension* $\leqq n$ *has injective dimension* $\leqq n$.

(5) *A direct limit of essential monomorphisms is an essential monomorphism.*

Proof. (1)⇒(3) is [6, Chapter I, Ex. 8].

(4)⇒(3) and (3)⇒(2) are trivial.

(2)⇒(1) is [7, Proposition 41].

(3)⇒(4) reduces easily to showing that a direct system has an injective resolution. To see that such resolutions exist we first take an injective module E containing, say, the direct sum of all Λ/I, I a left ideal in Λ. Then, for any Λ-module A, the obvious map $A \to E^{\mathrm{Hom}_\Lambda(A,E)}$ is a monomorphism into an injective module. (To the categorists, E is just an "injective generator.") The embedding above actually defines a natural transformation from the identity functor, and so it defines a similar embedding on any direct (or, for that matter, inverse) system(3).

(1)⇒(5) We are given a direct system $\{A_i, f_{ij}\}$ and a sub-direct system $\{B_i\}$, with B_i an essential submodule of A_i. If A and B denote the respective direct limits we must show that B is essential in A. For this it clearly suffices to show that if $\alpha \in A$ and $\Lambda\alpha \cap B = 0$ then $\alpha = 0$. If $f_i \colon A_i \to A$ is the natural map into the direct limit, then $\alpha = f_i(\alpha_i)$ for some i. Since Λ is left Noetherian we may assume, after choosing i suitably large that f_i is a monomorphism on $\Lambda\alpha_i$. Then since $f_i(\Lambda\alpha_i \cap B_i) \subset \Lambda\alpha \cap B = 0$ we have $\Lambda\alpha_i \cap B_i = 0$. Hence, since B_i is essential in A_i, $\Lambda\alpha_i = 0$, so $\alpha = f_i(\alpha_i) = 0$.

(5)⇒(1) If Λ is not left Noetherian choose a strictly increasing sequence $\{A_n\}$ of left ideals, and set $A = U_n A_n$. We define inductively an increasing sequence, $\{B_n\}$, of left ideals, satisfying: (i) $B_n \cap A = A_n$, and (ii) B_n/A_n is a maximal submodule of Λ/A_n having 0 intersection with A/A_n. The construction is a standard Zorn's lemma argument once it is observed that, if B_n has been chosen, then $B_n + A_{n+1}/A_{n+1} \cap A/A_{n+1} = 0$.

(3) This construction was pointed out to the author by Charles Watts.

Now with $B = U_n B_n$ we consider the direct system of modules Λ/B_n and submodules B/B_n. The direct limits are Λ/B and 0, respectively, so we shall have the required example once we show that B/B_n is essential in Λ/B_n for all n, and $B \neq \Lambda$. Viewing Λ/B_n as a quotient of Λ/A_n the first assertion follows from (ii) above and the general fact that if N is a submodule of M and K is maximal such that $K \cap N = 0$ then $N+K/K$ is essential in M/K (see [9]). Finally, since $B_n \cap A = A_n$ the chain $\{B_n\}$ is strictly increasing, so its union cannot equal Λ.

One of our principal uses of this theorem is for rings of quotients. We need the fact that this passage preserves minimal injective resolutions. The next lemma extends slightly an unpublished remark of Matlis.

Lemma 1.2. *Let R be a commutative ring, Λ an R-algebra, and S a multiplicatively closed set in R.*

(a) *If E is a Λ_S-module, then E is Λ-injective if and only if E is Λ_S-injective.*

(b) *If Λ is left Noetherian and E is Λ-injective, then E_S is both Λ- and Λ_S-injective.*

Proof. (a) If A is a finitely generated Λ-module, and E is a Λ_S-module, then

$$\mathrm{Hom}_\Lambda(A, E) = \mathrm{Hom}_\Lambda(A_S, E) = \mathrm{Hom}_{\Lambda_S}(A_S, E).$$

Since every finitely generated Λ_S-module has the form A_S as above, it follows that $\mathrm{Hom}_\Lambda(\ , E)$ is exact on finitely generated Λ-modules if and only if $\mathrm{Hom}_{\Lambda_S}(\ , E)$ is exact on finitely generated Λ_S-modules, and this, by [6, Chapter I, 3.2], establishes (a).

(b) Let E be Λ-injective. By (a) we need only show that E_S is Λ_S-injective. For a Λ-module A, the natural homomorphism

$$\mathrm{Hom}_\Lambda(A, E)_S \to \mathrm{Hom}_{\Lambda_S}(A_S, E_S)$$

is an isomorphism when A is finitely generated since Λ is left Noetherian. It follows then, as in (a) above, that $\mathrm{Hom}_{\Lambda_S}(\ , E_S)$ is exact on finitely generated Λ_S-modules, so E_S is Λ_S-injective.

Corollary 1.3. *Let Λ be a left Noetherian R-algebra and S a multiplicatively closed set in R. Then $R_S \otimes_R$ is an exact functor from Λ-modules to Λ_S-modules which preserves essential monomorphisms and injective modules, and, hence, minimal injective resolutions.*

Proof. The exactness is well known, and the preservation of injectives is contained in Lemma 1.2. It remains to show that if $0 \to B \to A$ is essential then so also is $0 \to B_S \to A_S$. Since A_S is a direct limit of modules $A_{S'}$, where $S' = \{x^n \mid n > 0\}$ for some $x \in R$, it follows from (5) of Theorem 1.1 that we may assume $S = \{x^n \mid n > 0\}$ for some $x \in R$. Let $A_n = A$ for all n and let $A_n \to A_{n+1}$ be multiplication by x. Then A_S is the direct limit of this system. Moreover, if $B_n = B \subset A_n$, this inclusion is essential, so the limit, B_S, of this sub-direct system, is essential in A_S, again by (5) of Theorem 1.1.

REMARK. This corollary fails, even when $\Lambda = R$, without the chain condition.

COROLLARY 1.4. *In the setting of Lemma* 1.2, *with* Λ *left Noetherian, if* A *is a* Λ*-module,* $\mathrm{Id}_{\Lambda_S}(A_S) \leqq \mathrm{Id}_\Lambda(A)$, *and* $\mathrm{Id}_\Lambda(A) = \sup \mathrm{Id}_{\Lambda_{\mathfrak{M}}}(A_{\mathfrak{M}})$, *where* $\mathfrak{M}$ *ranges over all maximal ideals of* R.

Proof. Let X be a minimal injective resolution of A. Then X_S is an injective resolution of A_S, by Lemma 1.2, and from this the first assertion follows. Now suppose $\mathrm{Id}_{\Lambda_{\mathfrak{M}}}(A_{\mathfrak{M}}) < n$ for all $\mathfrak{M}$. Then, since $X_{\mathfrak{M}}$ is a minimal resolution of $A_{\mathfrak{M}}$, by Corollary 1.3, we have $(X_n)_{\mathfrak{M}} = 0$ for all $\mathfrak{M}$, and it follows that $X_n = 0$ so $\mathrm{Id}_\Lambda(A) < n$. From this the second assertion follows.

2. A change of rings theorem.

LEMMA 2.1. *Let* $\mathfrak{A}$ *be a two sided ideal in* Λ. *Then* $\mathrm{Hom}_\Lambda(\Lambda/\mathfrak{A},\ )$ *is a left exact functor from* Λ*-modules to* $\Lambda/\mathfrak{A}$*-modules which preserves essential monomorphisms and injective modules.*

Proof. Half exactness is standard, and the statement about injectives is [6, Chapter II, 6.1a]. The assertion on essential monomorphisms is evident from the identification, $\mathrm{Hom}_\Lambda(\Lambda/\mathfrak{A}, A) = \{x \in A \mid \mathfrak{A}x = 0\}$.

THEOREM 2.2. *Let* Λ *be a left Noetherian* R*-algebra,* $x \in R$, *and* $S = \{x^r \mid r > 0\}$. *Assume* x *is a nonunit, nonzero divisor on* Λ, *and a nonzero divisor on the* Λ*-module* A. *Then* $\mathrm{Id}_\Lambda(A) < n$ *if and only if* $\mathrm{Id}_{\Lambda/x\Lambda}(A/xA) < n-1$ *and* $\mathrm{Id}_{\Lambda_S}(A_S) < n$.

Proof. By considering the exact sequence $0 \to \Lambda \xrightarrow{x} \Lambda \to \Lambda/x\Lambda \to 0$ we see that for any injective Λ-module E, $xE = E$. Let $0 \to A \to E_0 \to D \to 0$ be exact with E_0 an injective envelope of A. Since x is a nonzero divisor on A $\ker(E_0 \xrightarrow{x} E_0) \cap A = 0$ so $E_0 \xrightarrow{x} E_0$ is a monomorphism, hence an isomorphism, E_0 being an essential extension of A. This isomorphism induces $D = E_0/A \cong E_0/xA$. Again, since x is a nonzero divisor on E_0 we see that

$$\mathrm{Hom}_\Lambda(\Lambda/x\Lambda, E_0/xA) \cong \{e \in E_0 \mid xe \in xA\}/xA = A/xA,$$

so $\mathrm{Hom}_\Lambda(\Lambda/x\Lambda, D) \cong A/xA$. Now let

$$X = (0 \to D \to E_1 \to \cdots \to E_{n-1} \to E_n \to \cdots)$$

be a minimal injective resolution of D. Recall that $xE_0 = E_0$ so $xD = D$ and hence $xX = X$. It follows that the complex $\mathrm{Hom}_\Lambda(\Lambda/x\Lambda, X)$ is acyclic, so, by Lemma 2.1, it is a minimal $\Lambda/x\Lambda$-injective resolution of $\mathrm{Hom}_\Lambda(\Lambda/x\Lambda, D) \cong A/xA$.

We conclude, therefore, that $\mathrm{Id}(A) < n \Rightarrow E_n = 0 \Rightarrow \mathrm{Hom}_\Lambda(\Lambda/x\Lambda, E_n) = 0 \Rightarrow \mathrm{Id}_{\Lambda/x\Lambda}(A/xA) < n-1$. Moreover, by Corollary 1.4, $\mathrm{Id}_{\Lambda_S}(A_S) < n$, and this proves our theorem in one direction. Conversely, assume $\mathrm{Id}_{\Lambda/x\Lambda}(A/xA) < n-1$; then, from the construction above again, $\mathrm{Hom}_\Lambda(\Lambda/x\Lambda, E_n) = 0$, i.e. x is a nonzero divisor on E_n. Therefore $E_n \xrightarrow{x} E_n$ is an isomorphism so $E_n = (E_n)_S$.

Hence if, in addition, $\mathrm{Id}_{\Lambda_S}(A_S) < n$ we have $(E_n)_S = 0$ so $E_n = 0$ and $\mathrm{Id}(A) < n$.

3. **Self injective dimension.** As an immediate consequence of Theorem 2.2 we have:

COROLLARY 3.1. *If Λ is a left and right Noetherian ring with $\mathrm{Id}_\Lambda(\Lambda) \leqq 1$ and x is a central nonunit, nonzero divisor, then $\Lambda/x\Lambda$ is a quasi-Frobenius ring.*

Proof. We apply Theorem 2.2 with R the center of Λ and $A = \Lambda$, obtaining $\mathrm{Id}_{\Lambda/x\Lambda}(\Lambda/x\Lambda) \leqq 0$; i.e. $\Lambda/x\Lambda$ is left self injective. It follows then, by [10, Theorem 18], that $\Lambda/x\Lambda$ is quasi-Frobenius.

Our next theorem gives a characterization of rings of self injective dimension one which we shall require in [5]. Most of it was observed independently by Jans in [12], and we shall prove here only those portions not contained in Jans' paper.

If A is a Λ-module we write $A^* = \mathrm{Hom}_\Lambda(A, \Lambda)$, which is again a Λ-module (of the opposite kind). There is a natural homomorphism $\delta_A: A \to A^{**}$, and we call A *torsionless* if δ_A is a monomorphism, *reflexive* if δ_A is an isomorphism. A submodule B of A is said to be *closed in A* if A/B is torsionless.

LEMMA 3.2. *Let Λ be a left and right Noetherian ring and $0 \to A \to B \to C \to 0$ an exact sequence of finitely generated left Λ-modules with A and C reflexive. Then B is reflexive.*

Proof. By [6, Chapter V, §2] we can find a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F_A & \to & F_B & \to & F_C & \to & 0 \end{array}$$

with exact rows and columns, and with F_A, F_B and F_C free of finite rank (hence splitting the bottom row). Passing to duals we have a similar diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & A^* & \leftarrow & B^* & \leftarrow & C^* & \leftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \leftarrow & F_A^* & \leftarrow & F_B^* & \leftarrow & F_C^* & \leftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \leftarrow & K_A & \leftarrow & K_B & \leftarrow & K_C & \leftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where the K's are the appropriate cokernels; the exactness of the bottom row follows by diagram chasing. Now by [12, Theorem 1.1] A (resp. B, resp. C) is reflexive if and only if $\operatorname{Ext}^1_\Lambda(K_A, \Lambda)$ (resp. $\operatorname{Ext}^1_\Lambda(K_B, \Lambda)$, resp. $\operatorname{Ext}^1_\Lambda(K_C, \Lambda)$) $=0$. Therefore our conclusion follows from the exact sequence for Ext.

THEOREM 3.3 (JANS). *The following conditions are equivalent for a left and right Noetherian ring* Λ.

(1) $\operatorname{Id}_\Lambda(\Lambda) \leqq 1$ (*Λ viewed as a left Λ-module*).

(2) $\operatorname{Ext}^1_\Lambda(C, P)=0$ *whenever C and P are finitely generated left Λ-modules with C torsionless and P projective.*

(3) *A closed projective submodule of a finitely generated left Λ-module is a direct summand.*

(4) *Every torsionless finitely generated right Λ-module is reflexive.*

(5) *Every right ideal in Λ is reflexive.*

Proof. From Jans [12] we have (1)$\Leftrightarrow$(2) (Introduction), (2)$\Leftrightarrow$(4) (Theorem 1.1), and (1)$\Leftrightarrow$(4) (Corollary 1.3). Moreover, (2)$\Leftrightarrow$(3) by interpreting $\operatorname{Ext}^1_\Lambda$ as the group of extensions. Finally, to show that (4)$\Leftrightarrow$(5) we observe first [4, (4.5)] that a finitely generated right Λ-module A is torsionless if and only if $A \subset F$ for a free module F of finite rank. If rank $F=r$, then (5) is the case $r=1$ of (4). We prove (5)$\Rightarrow$(4) by induction on r, so assume $r>1$. Then $F=F_0\oplus F_1$ with F_0 and F_1 free of ranks $<r$. This decomposition induces an exact sequence $0\rightarrow A_0\rightarrow A\rightarrow A_1\rightarrow 0$ with $A_0=A\cap F_0$ and A_1 isomorphic to a submodule of F_1. By induction, then, A_0 and A_1 are reflexive, so the reflexiveness of A follows from Lemma 3.2.

COROLLARY 3.4. *If R is a Noetherian integral domain then* $\operatorname{Id}_R(R)\leqq 1$ *if and only if* $\mathfrak{A}=(\mathfrak{A}^{-1})^{-1}$ *for all nonzero ideals* $\mathfrak{A}$ *in* R.

Proof. If $\mathfrak{A}$ is a nonzero ideal, then $\mathfrak{A}^* \cong \mathfrak{A}^{-1}$, and, with this identification, $\delta_{\mathfrak{A}}$ is the inclusion of $\mathfrak{A}$ in $(\mathfrak{A}^{-1})^{-1}$. Thus the corollary is just (1)$\Leftrightarrow$(5) of the above theorem.

EXAMPLES. (1) It is shown in [5], on the basis of this corollary, that if R is an integral domain in which every ideal can be generated by two elements, then $\operatorname{Id}_R(R)\leqq 1$.

(2) If π is a finite group, and if R is a commutative ring with $\operatorname{Id}_R(R)=n$, then it follows from [10, Corollary 8] that $\operatorname{Id}_\Lambda(\Lambda)=n$, where $\Lambda=R\pi$. Integral group rings of finite groups are the prototype of rings described by Theorem 3.3.

4. **Relations with finitistic dimensions.** We shall adopt the following notation: if M is a Λ-module, $\operatorname{Pd}_\Lambda(M)$, $\operatorname{Wd}_\Lambda(M)$, and $\operatorname{Id}_\Lambda(M)$ denote the projective, weak, and injective dimensions respectively, of M. Moreover,

$$\text{left FPD}(\Lambda) = \sup \operatorname{Pd}_\Lambda(M),$$

$$\text{left FWD}(\Lambda) = \sup \operatorname{Wd}_\Lambda(M),$$

and

$$\text{left}\quad \mathrm{FID}(\Lambda) = \sup \mathrm{Id}_\Lambda(M),$$

where M ranges over all left Λ-modules for which, in each case, the designated dimension is finite.

PROPOSITION 4.1 (MATLIS, [14, Theorem 1]). *If Λ is left Noetherian, then left* $\mathrm{FID}(\Lambda) =$ *right* $\mathrm{FWD}(\Lambda)$.

PROPOSITION 4.2. *Suppose Λ is left Noetherian. If $\mathrm{Id}_\Lambda(\Lambda)$ and left $\mathrm{FID}(\Lambda)$ are both finite, they are equal.*

Proof. If left $\mathrm{FID}(\Lambda)=n$ choose a left Λ-module A with $\mathrm{Id}_\Lambda(A)=n$ and a module B with $\mathrm{Ext}^n_\Lambda(B, A)\neq 0$. Resolve A, $0\to K\to F\to A\to 0$, with F free. By Theorem 1.1 $\mathrm{Id}_\Lambda(F)=\mathrm{Id}_\Lambda(\Lambda)$ for any nonzero free module F. Hence, if $\mathrm{Id}_\Lambda(\Lambda)<\infty$ then $\mathrm{Id}_\Lambda(K)<\infty$, so $\mathrm{Ext}^{n+1}_\Lambda(B, K)=0$. Therefore the exact sequence,

$$\mathrm{Ext}^n_\Lambda(B, F)\to \mathrm{Ext}^n_\Lambda(B, A)\to \mathrm{Ext}^{n+1}_\Lambda(B, K)\quad \text{shows that}\quad \mathrm{Id}_\Lambda(F)\geqq n.$$

Since, by definition, $\mathrm{Id}_\Lambda(\Lambda)\leqq n$, we are done.

PROPOSITION 4.3. *If Λ is left Noetherian, then*

$$\textit{left}\ \mathrm{FPD}(\Lambda)\leqq \mathrm{Id}_\Lambda(\Lambda).$$

Proof. Suppose $\mathrm{Pd}_\Lambda(A)=n$. Choose a B for which $\mathrm{Ext}^n_\Lambda(A, B)\neq 0$, and resolve B, $0\to K\to F\to B\to 0$, with F free. Then $\mathrm{Ext}^{n+1}_\Lambda(A, K)=0$ so the exact sequence

$$\mathrm{Ext}^n_\Lambda(A, F)\to \mathrm{Ext}^n_\Lambda(A, B)\to \mathrm{Ext}^{n+1}_\Lambda(A, K)$$

shows that $\mathrm{Id}_\Lambda(F)\geqq n$. But, as in the proof above, $\mathrm{Id}_\Lambda(F)=\mathrm{Id}_\Lambda(\Lambda)$.

5. Finitistic dimension in commutative rings. R denotes henceforth a commutative Noetherian ring. Following [2] we call a sequence $a_1, \cdots, a_k$ of elements of R an *R-sequence* (or a *prime sequence* in [15] or [16]) if $(a_1, \cdots, a_{i-1}): a_i=(a_1, \cdots, a_{i-1})$, $i=1, \cdots, k$. The maximum length of an R-sequence is denoted Codim R, and the Krull dimension of R is written K-dim R. R is a *Macaulay-Cohen ring* (abbreviated *MC ring*) if Codim $R_{\mathfrak{M}}$ $=K$-dim $R_{\mathfrak{M}}$ for all maximal ideals $\mathfrak{M}$.

The results of this section, while concerning finitistic dimensions, have as their main consequence that if $\mathrm{Id}_R(R)<\infty$ then R is an MC ring. We first record some results we shall need.

PROPOSITION 5.1 (AUSLANDER-BUCHSBAUM).

(a) [3, Theorem 1.4] Codim $R=\sup$ Codim $R_{\mathfrak{M}}$, *where $\mathfrak{M}$ ranges over all maximal ideals.*

(b) [3, Theorem 2.4] $\mathrm{FWD}(R)=\sup$ Codim $R_{\mathfrak{P}}$, *where $\mathfrak{P}$ ranges over all prime ideals.*

The next proposition is a somewhat technical construction for which we have two important applications. The elements of the argument are present in [3, §2], though the conclusions we require cannot be derived directly from the results in [3].

PROPOSITION 5.2. *Suppose* K-dim $R \geqq n \geqq 1$. *Then there exist data*: (i) *a prime* $\mathfrak{P}$ *in* R; (ii) *an element* s *in* R; *and* (iii) *a sequence* $a_1, \cdots, a_{n-1}$ *in* $\mathfrak{P}$, *which satisfy* (a) K-dim $R/\mathfrak{P} \geqq 1$, (b) $\mathfrak{P} \neq (\mathfrak{P}, s) \neq R$, *and* (c) $a_1, \cdots, a_{n-1}$ *is both an* R_S- *and an* $R_{\mathfrak{P}}$-*sequence, where* $S = \{s^r \mid r > 0\}$.

Proof. We induce on n. If $n=1$ let $\mathfrak{P}$ be any nonmaximal prime of height 0 and s any element satisfying (b).

Suppose $n \geqq 2$. Choose a prime $\mathfrak{P}_1$ of height $n-1$ with K-dim $R/\mathfrak{P}_1 \geqq 1$. By induction in $R_{\mathfrak{P}_1}$ we produce a prime of $\mathfrak{Q} \subset \mathfrak{P}_1$ and a sequence $a_1, \cdots, a_{n-2}$ in $\mathfrak{Q}$ such that K-dim $R_{\mathfrak{P}_1}/\mathfrak{Q}R_{\mathfrak{P}_1} \geqq 1$ (so K-dim $R/\mathfrak{Q} \geqq 2$), and $a_1, \cdots, a_{n-2}$ is an $R_{\mathfrak{Q}}$-sequence; more precisely, the images of $a_1, \cdots, a_{n-2}$ in $R_{\mathfrak{Q}}$ are an $R_{\mathfrak{Q}}$-sequence. Now let $\mathfrak{P}_{ij}, j=1, \cdots, h_{\mathfrak{Q}}$ be those primes of $(a_1, \cdots, a_{i-1})$ containing a_i, $i=1, \cdots, n-2$, and let $\mathfrak{Q}_1, \cdots, \mathfrak{Q}_r$ be those primes $\mathfrak{Q}_k$ of $(a_1, \cdots, a_{n-2})$ for which ht $\mathfrak{Q}_k \geqq n-1$ and K-dim $R/\mathfrak{Q}_k \geqq 1$. Then $\mathfrak{P}_{ij} \not\subset \mathfrak{Q}$ for all i, j since $a_1, \cdots, a_{n-2}$ is an $R_{\mathfrak{Q}}$-sequence, and $\mathfrak{P}_1 \not\subset \mathfrak{Q}$ and $\mathfrak{Q}_k \not\subset \mathfrak{Q}$ for all k since ht $\mathfrak{Q} = n-2$. Consequently we may choose an element $t \in \mathfrak{P}_1 \cap \bigcap_{i,j} \mathfrak{P}_{ij} \cap \bigcap_k \mathfrak{Q}_k$, $t \notin \mathfrak{Q}$. Let $\mathfrak{M}$ be a maximal ideal of $R/\mathfrak{Q}$ containing $\mathfrak{P}_1/\mathfrak{Q}$. Then ht $\mathfrak{M} \geqq 2$, so $\mathfrak{M}$ contains infinitely many primes of height one, in particular one excluding the residue of t (recall $t \in \mathfrak{P}_1$, $t \notin \mathfrak{Q}$); this is an easy consequence of Krull's Hauptidealsatz (see [3, Proposition 2.6]). Lifting such a prime to a prime $\mathfrak{P}$ in R we have $\mathfrak{Q} \subsetneqq \mathfrak{P} \neq (\mathfrak{P}, t) \neq R$, so ht $\mathfrak{P} \geqq n-1$ and K-dim $R/\mathfrak{P} \geqq 1$. Therefore if $\mathfrak{P}$ belongs to the ideal $(a_1, \cdots, a_{n-2})$ then $\mathfrak{P} =$ some $\mathfrak{Q}_k$; but $t \in \mathfrak{Q}_k$ and $t \notin \mathfrak{P}$. Hence $\mathfrak{P}$ does not belong to $(a_1, \cdots, a_{n-2})$ so we can find an element $a_{n-1} \in \mathfrak{P}$ such that $(a_1, \cdots, a_{n-2})R_{\mathfrak{P}} : a_{n-1}R_{\mathfrak{P}} = (a_1, \cdots, a_{n-2})R_{\mathfrak{P}}$. Now, as above, we can choose an element $s' \notin \mathfrak{P}$ belonging to all primes of $(a_1, \cdots, a_{n-2})$ which contain a_{n-1}. Finally, it is straightforward to check that $\mathfrak{P}$, $s = s't$, and $a_1, \cdots, a_{n-1}$ satisfy our requirements.

COROLLARY 5.3. $\mathrm{FWD}(R) \leqq K$-dim $R \leqq \mathrm{FWD}(R)+1$. *If* R *is local* K-dim $R = \mathrm{FWD}(R)$ *if and only if* R *is an MC ring.*

Proof. The second statement follows immediately from the definition and Proposition 5.1(b). As for the first statement, the first inequality follows from Proposition 5.1(b), and for the second we may assume K-dim $R \neq 0$. Then if $\mathfrak{Q}$ is a prime ideal of height $n \geqq 1$ there is a prime $\mathfrak{P} \subset \mathfrak{Q}$ for which Codim $R_{\mathfrak{P}} = n-1$, by the proposition above, and from this our conclusion follows.

PROPOSITION 5.4. *In the setting of Proposition* 5.2,

$$\mathrm{Pd}_R(R_S/(a_1, \cdots, a_{n-1})R_S) = n.$$

Proof. Let $A_i = R_S/(a_1, \cdots, a_{i-1})R_S$, $i=1, \cdots, n$. Then $R_S = A_1$, $R_S/(a_1, \cdots, a_{n-1})R_S = A_n$, and the proposition asserts that $\mathrm{Pd}_R(A_n) = n$. We carry out the argument in three steps: (1) $\mathrm{Pd}_R(A_i) \leqq i$, $i=1, \cdots, n$; (2) If $\mathfrak{P}B=0$ then $\mathrm{Ext}_R^i(A_i, B) \cong \mathrm{Ext}_R^{i+1}(A_{i+1}, B)$, $i=1, \cdots, n-1$; and (3) For a suitable R-module B with $\mathfrak{P}B=0$ $\mathrm{Ext}_R^1(R_S, B) \neq 0$. It is clear that this will establish our result.

(1) Let $F=R[X]$, X an indeterminate, and let $G=(1-sX)R[X]$. Then F and G are R-free so, since $R_S \cong F/G$, $\mathrm{Pd}_R(R_S) \leqq 1$. Moreover, since $a_1, \cdots, a_{n-1}$ is an R_S-sequence, $\mathrm{Pd}_{R_S}(A_i) \leqq i-1$; this follows by induction from the exact sequences $0 \to A_i \to^{a_i} A_i \to A_{i+1} \to 0$. Finally, by a change of rings [6, Chapter XVI, Ex. 5], we have $\mathrm{Pd}_R(A_i) \leqq i$.

(2) Suppose $\mathfrak{P}B=0$. From the exact sequence $0 \to A_i \to^{a_i} A_i \to A_{i+1} \to 0$ we obtain

$$\mathrm{Ext}_R^i(A_i, B) \xrightarrow{a_i} \mathrm{Ext}_R^i(A_i, B) \to \mathrm{Ext}_R^{i+1}(A_{i+1}, B) \to \mathrm{Ext}_R^{i+1}(A_i, B).$$

By part (1) $\mathrm{Ext}_R^{i+1}(A_i, B)=0$. Since $a_i \in \mathfrak{P}$ and $\mathfrak{P}B=0$ the map on the left is 0. Thus $\mathrm{Ext}_R^i(A_i, B) \to \mathrm{Ext}_R^{i+1}(A_{i+1}, B)$ is an isomorphism.

(3) In the setting of part (1) put $B=G/\mathfrak{P}G$ and let $f\colon G \to B$ be the natural map. G is a free R-module with basis $Y_0 = 1-sX, \cdots, Y_n = X^n - sX^{n+1}, \cdots$, so B is a free $R/\mathfrak{P}$-module with basis $y_0 = f(Y_0), \cdots, y_n = f(Y_n), \cdots$. We wish to show that $\mathrm{Ext}_R^1(R_S, B) \neq 0$ so, by considering the exact sequence $\mathrm{Hom}_R(F, B) \to \mathrm{Hom}_R(G, B) \to \mathrm{Est}_R^1(R_S, B) \to 0$, it suffices to exhibit an element of $\mathrm{Hom}_R(G, B)$ which cannot be extended to F. We claim f is such a homomorphism.

For suppose there is a homomorphism $g\colon F \to B$ such that $g|G=f$. Let $x_n = g(X^n)$; then $y_n = x_n - sx_{n+1}$, $n=0, 1, \cdots$. From the first $n+1$ of these equations we have $x_0 - s^{n+1}x_{n+1} = y_0 + sy_1 + \cdots + s^n y_n$, or

$$(*) \qquad x_0 - y_0 - sy_1 - \cdots - s^n y_n = s^{n+1}x_{n+1}.$$

Now expand x_0 in terms of the basis $y_0, y_1, \cdots$; $x_0 = r_0 y_0 + \cdots + r_k y_k$ with $r_i \in R/\mathfrak{P}$. Then if $n > k$ the coefficient of y_n in $x_0 - y_0 - sy_1 - \cdots - s^n y_n$ is s^n. But, by (*), this coefficient is divisible by s^{n+1}. Since $R/\mathfrak{P}$ is an integral domain it follows that the residue of s is a unit in $R/\mathfrak{P}$; i.e. $(\mathfrak{P}, s) = R$. But this contradicts condition (b) of Proposition 5.2.

COROLLARY 5.5.

$$\mathrm{Codim}\, R \leqq \mathrm{FWD}(R) = \mathrm{FID}(R) \leqq K\text{-}\dim R \leqq \mathrm{FPD}(R) \leqq \mathrm{Id}_R(R) \leqq \mathrm{gl.\,dim}\, R.$$

Proof. From left to right the relations follow, respectively, from Proposition 5.1, Proposition 4.1, Proposition 5.1 again (and Krull's "prime ideal theorem"), Proposition 5.4 above, Proposition 4.3, and, finally, the definitions.

COROLLARY 5.6. *If* $\mathrm{Id}_R(R) < \infty$ *then*

$$\mathrm{Id}_R(R) = \mathrm{Codim}\ R = \mathrm{FWD}(R) = \mathrm{FID}(R) = K\text{-dim}\ R = \mathrm{FPD}(R).$$

Proof. Except for Codim R this is an immediate consequence of the corollary above. The insertion of Codim R is then permitted by Corollary 5.3, since the latter is computed locally.

REMARK. We know of no ring R for which K-dim $R \neq \mathrm{FPD}(R)$.

6. **Self injective dimension in commutative rings.** Before proceeding to our main theorem we shall recall some elementary facts about irreducible submodules and ideals. If M is a finitely generated R-module and Q a submodule we call Q *irreducible* (in M) if Q is not the intersection of two larger submodules. If $\mathfrak{P}$ is a prime in R, Q is called $\mathfrak{P}$-*primary* if $(Q:\alpha)$ is a $\mathfrak{P}$-primary ideal for all $\alpha \in M$, $\alpha \notin Q$. The following lemma is well known.

LEMMA 6.1. *If Q is a proper submodule of a finitely generated R-module M, then Q is irreducible in M if and only if Q is $\mathfrak{P}$-primary for some prime $\mathfrak{P}$ and $Q_{\mathfrak{P}}$ is irreducible in $M_{\mathfrak{P}}$.*

Now let N be any proper submodule in M. Then we can write $N = \bigcap_i Q_i$ with each Q_i irreducible and with no inclusions among the Q_i. If $\mathfrak{P}$ is a fixed prime ideal let r denote the number of Q_i which are $\mathfrak{P}$-primary. It follows then from the structure theory of Matlis in [13] that if $E(M/N)$, the injective envelope of M/N, is written as a direct sum of indecomposable modules, then precisely r of the direct summands are isomorphic to $E(R/\mathfrak{P})$. Hence r is independent of the representation of N as above.

Alternatively, we can interpret r as follows: If we call the sum of all simple submodules of a module its *socle*, then r is the dimension over $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$ of the socle of the $R_{\mathfrak{P}}$-module $M_{\mathfrak{P}}/N_{\mathfrak{P}}$.

PROPOSITION 6.2. *Suppose R is a local Artinian ring with maximal ideal $\mathfrak{M}$, and $K = R/\mathfrak{M}$. Then R is a quasi-Frobenius ring if and only if the socle of R has dimension one over K; i.e. if and only if* (0) *is irreducible in R.*

Proof. By [8, Theorems 3.4 and 4.2] R is a quasi-Frobenius ring if and only if $M = M^{**}$ for all finitely generated modules M. Now $K^* = \mathrm{Hom}_R(R/\mathfrak{M}, R)$ is just the socle of R, and K^* is isomorphic to a direct sum of copies of K. It is thus clear that $K = K^{**}$, if and only if the socle is one dimensional. It therefore remains only to show that if $K = K^{**}$, and if M is any finitely generated module, then $M = M^{**}$. But an obvious induction on length reduces this assertion to Lemma 3.2.

Let R be a local MC ring, and let $\mathfrak{Q}$ be the ideal generated by a system of parameters (s.o.p.). Then it follows from [15, Theorem 1.3] of [16, Appendix 6, Lemma 3] and the remarks above that the dimension, n, of the socle of $R/\mathfrak{Q}$—i.e. the number of irreducible components of $\mathfrak{Q}$—is independent of the s.o.p. chosen. While this invariance is not valid for arbitrary local rings, neither does it characterize those which are MC rings. In any event we shall refer to an MC ring R as above, more specifically, as an *MCn ring*. Thus an *MC*1 ring is a local MC ring in which every s.o.p. generates an irreducible ideal. The theorem of Northcott and Rees referred to above generalizes a theorem of Grobner [11] which asserts that regular local rings are MC1. In general, if R is MCn, and if $\mathfrak{A}$ is generated by an R-sequence, then $R/\mathfrak{A}$ is again MCn. Thus, any local complete intersection (regular local ring modulo an R-sequence) is MC1. A local Artinian ring R(K-dim$=0$) is MCn, where n is the dimension of the socle of R.

MC rings are characterized by the fact that an ideal $\mathfrak{A}$ generated by an R-sequence of length k is unmixed of height k (see [15] or [16, Appendix 6]). We call $\mathfrak{A}$ *irreducibly unmixed of height* k if, in addition, the primary components of $\mathfrak{A}$ are irreducible.

THEOREM 6.3. *The following are equivalent for a ring R of finite Krull dimension.*

(1) $\mathrm{Id}_R(R)<\infty$.

(2) $\mathrm{Id}_R(R)=K\text{-dim } R=\mathrm{FID}(R)$.

(3) $\mathrm{Id}_{R/\mathfrak{A}}(R/\mathfrak{A})<\infty$ *whenever* $\mathfrak{A}$ *is generated by an R-sequence.*

(4) $R_{\mathfrak{P}}$ *is an MC*1 *ring for all prime ideals* $\mathfrak{P}$.

(5) *If an ideal* $\mathfrak{A}$ *of height k is generated by k elements then* $\mathfrak{A}$ *is irreducibly unmixed of height k.*

Proof. (1)$\Leftrightarrow$(2) is contained in Corollary 5.6. (3)$\Rightarrow$(1) by taking $\mathfrak{A}=0$, and (1)$\Rightarrow$(3) is an immediate consequence of Theorem 2.2.

(3)$\Rightarrow$(4). Since $\mathrm{Id}_R(R)\geqq\mathrm{Id}_{R_{\mathfrak{P}}}(R_{\mathfrak{P}})$ (Corollary 1.4) we may as well assume $R=R_{\mathfrak{P}}$, so R is local. Then Corollary 5.6 tells us that Codim $R=K$-dim R, so R is an MC ring. Therefore an ideal $\mathfrak{Q}$ generated by a s.o.p. is generated by an R-sequence, so, by (3), $\mathrm{Id}_{R/\mathfrak{Q}}(R/\mathfrak{Q})<\infty$. Since K-dim $R/\mathfrak{Q}=0$ it follows from Corollary 5.6 again that $\mathrm{Id}_{R/\mathfrak{Q}}(R/\mathfrak{Q})=0$, so $R/\mathfrak{Q}$ is a quasi-Frobenius ring, by [10, Theorem 18]. Hence by Proposition 6.2, $\mathfrak{Q}$ is irreducible.

(4)$\Rightarrow$(1). Assuming $R_{\mathfrak{P}}$ is MC1 for all primes $\mathfrak{P}$ we shall prove $\mathrm{Id}_R(R)<\infty$ by induction on K-dim R. By Corollary 1.4 it suffices to show that $\mathrm{Id}_{R_{\mathfrak{P}}}(R_{\mathfrak{P}})<\infty$ for all primes $\mathfrak{P}$. If ht $\mathfrak{P}=0$ this follows from Proposition 6.2 and [10, Theorem 18]. Otherwise we may choose a nonzero divisor $x\in\mathfrak{P}R_{\mathfrak{P}}$. Then if $S=\{x^r \mid r>0\}$ both $R_{\mathfrak{P}}/xR_{\mathfrak{P}}$ and $(R_{\mathfrak{P}})_S$ inherit our hypothesis, clearly. Hence, by induction, $\mathrm{Id}_{R_{\mathfrak{P}}/xR_{\mathfrak{P}}}(R_{\mathfrak{P}}/xR_{\mathfrak{P}})<\infty$ and $\mathrm{Id}_{(R_{\mathfrak{P}})_S}((R_{\mathfrak{P}})_S)<\infty$.

Now by Theorem 2.2 $\mathrm{Id}_{R_{\mathfrak{P}}}(R_{\mathfrak{P}})<\infty$.

(4)$\Rightarrow$(5). Given $\mathfrak{A}$ as in (5), $\mathfrak{A}$ is unmixed of height k by [15, Theorem 2.2]. If we localize at a prime $\mathfrak{P}$ of $\mathfrak{A}$, then $\mathfrak{A}R_{\mathfrak{P}}=\mathfrak{Q}R_{\mathfrak{P}}$, where $\mathfrak{Q}$ is the $\mathfrak{P}$-primary component of $\mathfrak{A}$. Hence $\mathfrak{Q}R_{\mathfrak{P}}$ is generated by a s.o.p. so $\mathfrak{Q}R_{\mathfrak{P}}$ is irreducible.

(5)$\Rightarrow$(4). (4) is just a special case of (5) once we observe, using Lemma 6.1, that (5) is inherited by every $R_{\mathfrak{P}}$.

As pointed out in the introduction we can now conclude that, for local rings: regular $\Rightarrow$ local complete intersection $\Rightarrow$ finite self injective dimension $\Rightarrow$MC1$\Rightarrow$MC. This indicates a number of examples to which Theorem 6.3 applies. We might note further that, while local complete intersections are defined to be quotients of regular local rings, one can define them in general using Assmus' characterization [1, Theorem 2.7]. It follows then, using Theorem 2.2, that local complete intersections, even in this more general sense, have finite self injective dimension. The following example which goes back to Macaulay, and which was pointed out to the author by D. G. Northcott, shows that local complete intersections do not exhaust the local rings of finite self injective dimension: With K a field,

$$R = K[[x, y, z]]/(x^2 - y^2, y^2 - z^2, xy, yz, zx)$$

is a quasi-Frobenius ring, but manifestly not a local complete intersection.

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