A SINGULAR PERTURBATION METHOD. PART II*

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Abstract. This paper extends the work of Part I to the partial differential equation

$$\epsilon^3\nabla^2\psi - g(\mathbf{x})\psi = 0$$

where ϵ is a small positive parameter and $g(\mathbf{x})$ is a bounded function of $\mathbf{x}$ which vanishes along simple closed surfaces in the solution domain. In particular, the eigenproblem corresponding to the case in which $g(\mathbf{x})$ is positive at infinity and in which the boundary condition $\psi \to 0$ as $|\mathbf{x}| \to \infty$ is imposed, is considered. One class of eigensolutions is extracted.

1. Introduction. The ideas which have been developed by consideration of linear ordinary differential equations in Part I are now applied to an eigenvalue problem associated with the partial differential equation,

$$\epsilon^3\nabla^2\psi - g(\mathbf{x})\psi = 0, \tag{1.1}$$

where $\mathbf{x}$ is a vector in m-dimensional Euclidean space, and ϵ is a small positive parameter. The case will be considered where $g(\mathbf{x})$ is analytic,[1] positive at infinity, and negative only in a simply connected domain, D, bounded by a closed surface Γ. Hereafter, the surface Γ will be referred to as the *transition surface*, to correspond with the terminology, *transition point*, in the one-dimensional case. The boundary condition, $\psi \to 0$ as $|\mathbf{x}| \to \infty$, is imposed, and it is required to find those values of ϵ for which a nontrivial solution to this problem exists. This problem is closely related to the problem of finding the highly excited bound states of the Schrodinger wave equation potential well problem.

There is no need to stipulate that there is a *single* closed transition surface for, if there are several, the solution can be considered as in the case of ordinary differential equations, as the union of solutions in separate regions matched across suitably determined boundaries. It will become apparent to the reader when the form of the present solution is obtained that the other restrictions imposed place no essential limitation on the method. Thus, for example, if g is negative at infinity and suitable boundary conditions are imposed on the solution, the same method can be employed to solve the equation.

If $(-g(\mathbf{x}))^{1/2}$ is thought of as the variable refractive index of a medium, and $(\epsilon^{-2/3})$ as the propagation constant which is proportional to the angular frequency of the field, then the solution, $\psi(\mathbf{x})$, of Eq. (1.1), represents the standing wave pattern set up in a medium bounded by the reflexion surface Γ. The propagation constant $(\epsilon^{-2/3})$ is large, so the problem is a high frequency propagation problem, and therefore, it may be expected that many of the results to be obtained will be closely linked with the results of geometric

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[1]This condition may be relaxed considerably.

optics. Indeed, it will be seen that the rays and wavefronts of geometric optics play an important role in the analysis to follow.

Although the story is quite complicated, the eventual solution technique is straightforward and is summarized in the conclusion.

2. Procedural motivation. The same approach is employed here as was employed in the simpler one-dimensional case, described in Part I. It will be seen that there is a very close analogy to be drawn between these two cases, and for this reason, only a brief description will be included here of the ideas motivating the formal procedure as described in the next section.

As in the one-dimensional case, a description of the solution behaviour close to the transition surface if first sought. In order to do this the equation is described in terms of a co-ordinate system centered on the transition surface. Any co-ordinate system $(\tau, \mathbf{s})$, where τ determines distance from the transition surface and $\mathbf{s}$ specifies, uniquely, position around the transition surface in its neighbourhood, may be employed for this purpose, because this system is introduced only in order to develop a suitable mathematical form for a uniformly valid solution. If the transition surface is suitably regular, the equation for ψ in terms of this new co-ordinate system will be of the form

$$\epsilon^3(\psi_{\tau\tau} + \nabla\psi_\tau\cdot\mathbf{a} + \nabla\cdot\nabla\psi + \nabla\psi\cdot\mathbf{b} + \psi_\tau c) - \psi\sum_0^\infty g_k(\mathbf{s})\tau^{k+1} = 0, \qquad (2.1)$$

where $\mathbf{a}$, $\mathbf{b}$ and c are analytic in $\mathbf{s}$ and τ close to Γ, and the symbol ∇ denotes the gradient with respect to $\mathbf{s}$, τ remaining constant.

If nontrivial solutions to this equation exist, they must vary rapidly in some or all of the $(\tau, \mathbf{s})$ directions. It seems reasonable to commence by looking for solutions slowly varying around the transition surface, Γ. In this case, the dominating terms of the above Eq. (2.1) will be $\epsilon^3\psi_{\tau\tau}$, and $g_0(\mathbf{s})\tau\psi$, and a balance will be achieved between these two terms if ψ varies by unit order in a range τ of order ϵ. This suggests the introduction of the stretched co-ordinate $\eta = \tau/\epsilon$, in terms of which Eq. (2.1) for ψ becomes

$$\psi_{\eta\eta} - g_0(\mathbf{s})\eta\psi = \text{terms in } \epsilon, \epsilon^2, \qquad (2.2)$$

which is analogous to the equation obtained in the one-dimensional case of Part I. Again, as in the one-dimensional case, a normal perturbation determination for ψ of the form, $\psi = \psi_0(\eta) + \epsilon\psi_1(\eta) + \cdots$, runs into difficulty because the terms on the right hand side of Eq. (2.2) for ψ throw up terms in ψ_η which do not fit into the asymptotic scheme for large η. To overcome this, heuristic ideas analogous to those employed in the simpler case, *suggest* that an asymptotic solution of the form $\psi = \psi(\zeta, \mathbf{s}, \tau, \epsilon)$, or, in terms of $\mathbf{x}$, $\psi = \psi(\zeta, \mathbf{x}, \epsilon)$, where $\zeta = \epsilon^{-1}u(\mathbf{x})$, be sought.

It should be noted that *this solution form suggests itself, only when solutions varying slowly around the transition surface* Γ, *are being considered.* By seeking solutions varying rapidly in certain tangential directions around Γ, it may be possible to extend the class of solutions considered. This, however, will not be done here.

3. The formal solution. In this section a formal substitution of the solution form $\psi = \psi(\zeta, \mathbf{x}, \epsilon)$, suggested by the work of the previous section, is carried out. In terms of the $(\zeta, \mathbf{x})$ co-ordinate system,

$$\nabla\psi(\mathbf{x}, \epsilon) = \epsilon^{-1}\psi_\zeta(\mathbf{x}, \zeta, \epsilon)\nabla u(\mathbf{x}) + \nabla\psi(\mathbf{x}, \zeta, \epsilon)$$

and,

$$\nabla^2\psi(\mathbf{x}, \epsilon) = \epsilon^{-2}\{\psi_{\zeta\zeta}(\mathbf{x}, \zeta, \epsilon)[\nabla u(\mathbf{x})]^2\} + \epsilon^{-1}\{2\nabla\psi_\zeta(\mathbf{x}, \zeta, \epsilon)\cdot\nabla u(\mathbf{x}) + \psi_\zeta(\mathbf{x}, \zeta, \epsilon)\nabla^2 u(\mathbf{x})\} + \{\nabla^2\psi(\mathbf{x}, \zeta, \epsilon)\},$$

where $\nabla\psi(\mathbf{x}, \zeta, \epsilon)$ denotes the gradient of ψ with respect to $\mathbf{x}$, ζ remaining constant. A substitution of this expression in Eq. (1.1) results in the equation for ψ in the form

$$\psi_{\zeta\zeta} - \zeta(g/[u(\nabla u)^2])\psi = -1/(\nabla u)^2\{\epsilon(2\nabla\psi_\zeta\cdot\nabla u + \psi_\zeta\nabla^2 u) + \epsilon^2\nabla^2\psi\}, \tag{3.1}$$

where, as in the one-dimensional case, ϵ^{-1} is replaced by ζ/u.

This result is rather surprising because it suggests that the partial differential equation (1.1) for ψ can be dominated by the ordinary differential equation (3.1). It should be remembered, however, that the previous heuristic development applies *only* to those solutions which *vary slowly around the transition surface.*

This equation is formally equivalent (with the derivative, d/dx, which occurs on the ordinary differential equation case, being replaced by the vector operator ∇) to that obtained in the simpler case of Part I. One can proceed formally as in the one-dimensional case, or simply recognize that the two cases are formally equivalent and look for solutions of the form

$$\psi = A(\mathbf{x}, \epsilon)Ai(\zeta) + \epsilon^2 B(\mathbf{x}, \epsilon)Ai'(\zeta) \tag{3.2}$$

where the "boundary layer co-ordinate"[2] $u(\mathbf{x})$ satisfies

$$u(\nabla u)^2 = g(\mathbf{x}) \tag{3.3}$$

with $u = 0$ along Γ. Direct substitution leads to the following equations for $A(\mathbf{x}, \epsilon)$ and $B(\mathbf{x}, \epsilon)$,

$$2(\nabla A\cdot\nabla u) + \nabla^2 uA = -\epsilon^3\nabla^2 B \tag{3.4}$$

and

$$2u(\nabla B\cdot\nabla u) + uB\nabla^2 u + (\nabla u)^2 B = -\nabla^2 A, \tag{3.5}$$

which are again formally equivalent to the equations obtained by direct substitution in the one-dimensional case of Part I.

It is clear from the earlier work that it is the real solution of the nonlinear hyperbolic boundary layer Eq. (3.3) that is required here. The useful property that the *characteristics* of this equation are the *orthogonal trajectories of the level curves* $u = const$ follows directly from the characteristic form of Eq. (3.3). Although the characteristic equations for u cannot be explicitly integrated, they can be integrated numerically in any particular case. The question of whether or not Eq. (3.3) has a well-behaved solution does however arise, and this question will be discussed later in Sec. 5 when more is known of the significance of the "boundary layer" function.

In the hope that the form suggested by the heuristic arguments of Sec. (2) might lead to an evaluation, or at least a simplification, of Eq. (1.1) for ψ, certain formal manipulations, resulting in the relationships (3.2), (3.3), (3.4), and (3.5), have been performed. The question therefore to be asked now is—"Does the form (3.2) suggested by the heuristic arguments lead to a straightforward perturbation problem, (as defined by Eqs. (3.3), (3.4), and (3.5)) or have the transformations so far performed merely led to equations whose solutions are at least equally singular in their dependence on

[2]This corresponds to the terminology used in the one-dimensional case.

ϵ?" In order to answer this question, a close investigation of the equations for A and B has to be undertaken.

For the sake of simplicity and ease in diagrammatic representation in the work to follow, the two-dimensional case will be considered. It will be obvious to the reader that the underlying ideas in the following are by no means restricted to this case; and that an extension of the method to higher dimensional spaces involves nothing essentially different. In the two-dimensional case the transition surface Γ degenerates to a line, which will be referred to as the "transition line".

4. An iterative procedure for determining A and B. The form of Eqs. ((3.4) and (3.5)) determining A and B, suggests that asymptotic expansions of the form

$$A = \sum_0^\infty \epsilon^{3k} A_k(\mathbf{x}), \qquad \text{and} \qquad B = \sum_0^\infty \epsilon^{3k} B_k(\mathbf{x}), \tag{4.1}$$

be sought. A formal substitution of these expressions for A and B in Eqs. (3.4) and (3.5), yields

$$2(\nabla A_k \cdot \nabla u) + \nabla^2 u A_k = -\nabla^2 B_{k-1} \tag{4.2}$$

and

$$2u(\nabla B_k \cdot \nabla u) + uB_k \nabla^2 u + (\nabla u)^2 B_k = -\nabla^2 A_k \,, \tag{4.3}$$

for all $k \geq 0$, where it is understood that $B_{-1} = 0$. An iterative procedure can now be employed to determine A_k and B_k for successive values of k. The equations determining A_k and B_k are all first order, linear, hyperbolic differential equations, the characteristics of which coincide with the characteristics of the equation for u. These *characteristics therefore form the "rays" of propagation* (using the terminology of geometric optics) of the solution $\psi(\mathbf{x})$. Also, the *orthogonal trajectories of the level u curves on the* (x, y) *plane* are perpendicular to the shared characteristics, and therefore *mark off on the* (x, y) *plane the "wavefronts"* of the solution $\psi(\mathbf{x})$. In order to obtain a clear picture of the solution properties, it is necessary to change to a new (characteristic) co-ordinate system based on these rays and wavefronts. This new co-ordinate net is defined by

(i) $\alpha = -u$ and,

(ii) β is constant along any characteristic, and the value of β uniquely determines this characteristic.

Thus, for the moment, the particular choice of β is kept open and disposable at convenience. Equations (4.2) and (4.3) when referred to this co-ordinate net reduce to their simplest forms,

$$(A_k S^{1/2})_\alpha = +1/(2S^{1/2})\gamma(B_{k-1}), \tag{4.4}$$

and

$$(B_k(\alpha S)^{1/2})_\alpha = -1/(2(\alpha S)^{1/2})\gamma(A_k), \tag{4.5}$$

where γ is a partial differential operator defined by

$$\gamma(\omega) = \frac{\partial}{\partial \alpha}\left(S \frac{\partial \omega}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{S}\frac{\partial \omega}{\partial \beta}\right), \tag{4.6}$$

and

$$S(\alpha, \beta) = h_2/h_1 , \tag{4.7}$$

where h_1 and h_2 are scaling factors, corresponding to the new co-ordinate net, defined by

$$ds^2 = h_1^2\, d\alpha^2 + h_2^2\, d\beta^2. \tag{4.8}$$

In an iterative determination of A_k and B_k Eqs. (4.4) and (4.5) become, effectively, ordinary linear differential equations in α, which can be integrated to give as general solutions,

$$A_k = \frac{F_k(\beta)}{S^{1/2}} + \int_0^\alpha \frac{\gamma(B_{k-1})}{2S^{1/2}}\, d\alpha,$$

and

$$B_k = -\frac{1}{(\alpha S)^{1/2}} \int_0^\alpha \frac{1}{2(\alpha S)^{1/2}}\, \gamma(A_k)\, d\alpha + \frac{G_k(\beta)}{(\alpha S)^{1/2}} ,$$

where $F_k(\beta)$ and $G_k(\beta)$ are, at present, arbitrary functions of β. Clearly, to ensure the continuity of B_k and A_k close to the transition line Γ (where α vanishes (see Eq. (3.3)), it is necessary to put $G_k(\beta) = 0$, as in the ordinary differential equation case of Part I.[3] The boundedness conditions on the derivatives of B_k and A_k with respect to α near $\alpha = 0$ are also ensured by this condition, as can be seen by comparison with the discussion in the ordinary differential equation case. The solutions for A_k and B_k now become

$$A_k = \frac{F_k(\beta)}{S^{1/2}} + \int_0^\alpha \frac{\gamma(B_{k-1})}{2(S)^{1/2}}\, d\alpha, \tag{4.9}$$

and

$$B_k = -\frac{1}{(\alpha S)^{1/2}} \int_0^\alpha \frac{1}{2(\alpha S)^{1/2}}\, \gamma(A_k)\, d\alpha, \tag{4.10}$$

for all $k \geq 0$, where B_{-1} is interpreted to be zero, and where the $F_k(\beta)$, for all $k \geq 0$, *are still arbitrary functions of* β. In particular,

$$A_0 = F_0(\beta)/S^{1/2}. \tag{4.11}$$

The formal equivalence of the results obtained here and the results obtained in Part I *suggests* that the expansion,

$$\psi = Ai(\zeta) \sum_0^\infty \epsilon^{3k} A_k(\alpha, \beta) + \epsilon^2 Ai'(\zeta) \sum_0^\infty \epsilon^{3k} B_k(\alpha, \beta) \tag{4.12}$$

(which follows from (3.2) and (4.1)) is *indeed an asymptotic representation of the desired solution except* possibly in isolated regions of the solution domain *where S vanishes or is singular.* It will be seen (in Sec. 6) that *it is the isolated points of misbehaviour of S* in the solution domain which *determine the arbitrary functions of* β still present in the solution expansion.

NOTE. It is particularly significant that the approach presented above leads one quite straightforwardly into the "natural" co-ordinate net of this problem—the net based on the rays and wavefronts.

5. Mapping singularities. It has been established that any iterative solution of Eqs. (3.4) and (3.5) for A and B is closely associated with the properties of the level

[3] $\gamma(A_k)$ and $\gamma(B_{k-1})$ are regular at $\alpha = 0$ once this condition is imposed.

curves (the wave fronts) of u and their orthogonal trajectories (the rays). A satisfactory determination of A and B over the whole field by this method is thus dependent on the existence of a satisfactory solution for u at every point in the field. In particular, a single-valued determination of u at each point in the field is required.

At a particular point $\mathbf{x}$, the boundary layer function $u(\mathbf{x})$ is evaluated solely in terms of the characteristic (i.e. "ray") passing through this point so that a multiple u coverage of the (x, y) plane can only be realized in regions of overlap of these characteristics in the (x, y) plane. The transition line Γ is closed, and therefore, at least within Γ, it is inevitable that characteristics will overlap and a multiple coverage of the (x, y) plane will result.[4] There may also be regions in the (x, y) plane not crossed by characteristics within which, therefore, no u coverage has yet been achieved. Since the characteristics form the "rays" of propagation, these regions by their very nature are simply "*shadow*" regions so that this situation need not be of concern. A closer examination of the implications of Eqs. (3.3) and (4.7) confirms that the solution $\psi(\mathbf{x})$, as given by (3.2) indeed drops to zero on passing into such a shadow region.[5]

Some typical overlap situations are now envisaged. The first that springs to mind is the focusing situation in which a band of characteristics converges on and passes through a single point—*the focus*. e.g. in the centrally symmetric case $g(\mathbf{x}) = g(r)$ characteristics focus on $r = 0$. The characteristics may overlap to form *caustics*, a situation which is illustrated in Fig. 1. Variants of this situation which require no special attention do arise. One such variant is illustrated in Fig. 2.

The fact that characteristics overlap so that the (x, y) plane is multiply covered does not prevent use of the (α, β) co-ordinate net in the present problem, however, if the solution (3.2) is to have any meaning, a single sheet coverage must be defined. In the single focus case, the obvious way to produce a single sheet coverage of the (x, y) plane

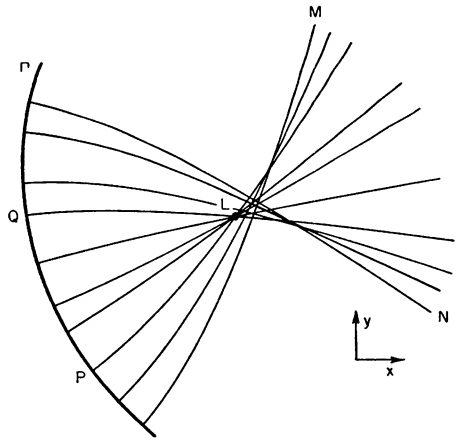

FIG. 1 CAUSTICS

FIGURE 1

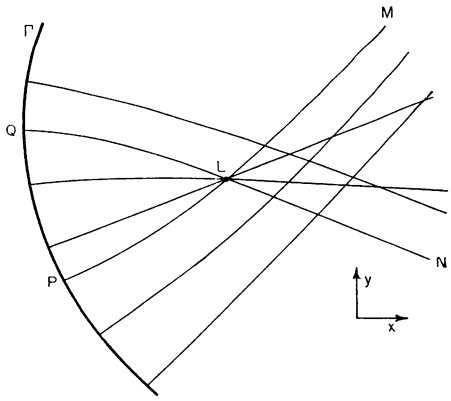

FIG. 2

FIGURE 2

[4]It is clear also from the fact that the equation determining u is *hyperbolic* and the boundary condition $u = 0$ is to be satisfied along a *closed* curve Γ that a unique solution will not result.

[5]See [2] for a more detailed discussion.

is to stop all characteristics once they have reached the focus. Even so the mapping remains singular at the focus. It will be seen later (in Sec. 7) that it is this singularity in the mapping at the focus that restricts the solution behaviour in such a way that the unknown $F_0(\beta)$ in the expansion for $\psi(\mathbf{x})$ (see (4.11)) can be determined. In Figs. 1 and 2 above, clearly, to produce a single sheet coverage of the (x, y) plane it is necessary to cut from one solution sheet to another, somewhere between LM and LN. There are, of course, many ways of cutting from one sheet to another to produce a single-valued coverage of the (x, y) plane. It is important, however, to notice that, although this cutting procedure can be defined in many different ways, the end-point L, of the cut is fixed in the (x, y) plane.

Aside. It may seem strange that a certain amount of arbitrariness is associated with the positioning of this cut in the (x, y) plane. It should be remembered, however, that in the one-dimensional case of Part I, a similar situation arose. There it was found that the matching point (corresponding to the cut, in the more general case under consideration here) could be placed anywhere between the two transition points excluding a small neighbourhood of these points. As in the one-dimensional case, therefore, it might be expected that different procedures for introducing the cut produce simply different asymptotic representations of the same solution. This point is taken up again in Sec. 8.

It is clear from the above, that by introducing suitable "cuts" in the solution domain a single valued coverage can be achieved (except along "cuts" and at foci) for any $g(\mathbf{x})$ and that *many different* cutting arrangements can by employed to do just this. It is also clear from the above that, having adopted one of these "cut configurations", one *at most* has to *cope* with *a number* of *"cuts" and "foci" in the solution domain.* In order to be explicit a procedure which leads to a *particular cut configuration* will now be described, and this configuration will then be adopted for the analysis of the next two sections where the solution behaviour along characteristics emanating from cuts (Sec. 6) and foci (Sec. 7) will be examined. The question of the equivalence of the various cut configurations will be discussed in section 8.

Along the transition line, Γ, u is required to vanish. The characteristics are orthogonal to the level u curves so they are "initially" orthogonal to Γ. There will be a smallest value of $|u|$ at which at least two characteristics intersect (e.g. $|u(L)|$ in Figs. 1 and 2). For values of $|u|$ less than this value the characteristic equations for u lead to $u(x, y)$ as a single-valued function of position. For larger values there is no unique determination and one has to decide on which characteristics to base further evaluations (e.g. beyond L in Figs. 1 and 2 characteristics cross and there is no unique determination). The decision is made to make further determinations on the basis of characteristics other than those that have so far intersected (e.g. other than the characteristics intersecting at L) and furthermore, as one integrates forward in $|u|$ to ignore characteristics for further evaluations once they have intersected. The cut is, of course, the locus of the intersection points. The area of the solution domain left uncovered is continuously reduced as one integrates forward in $|u|$ so that eventually this process leads to a complete coverage. Notice that this particular cutting procedure leads to a co-ordinate net in which u is continuous across cuts.

Some simple illustrative examples are now discussed. In the case in which $g(x, y) = a^2x^2 + b^2y^2 + c^2$ symmetry arguments indicate that the level u curves are concentric ellipses so that all characteristics focus on $x = y = 0$. A more interesting situation arises when there are two or more foci within Γ—e.g. in the case in which

$$g(x, y) = a + \frac{b}{y^2} - \frac{1}{[(x-1)^2 + y^2]^{1/2}} - \frac{1}{[(x+1)^2 + y^2]^{1/2}}, \tag{5.1}$$

which is closely related to the helium molecule ion problem, there are two foci corresponding to the two neuclei at $x = \pm 1 y = 0$. An application of the above prescribed procedure leads to the "cut configuration" indicated in Fig. 3.

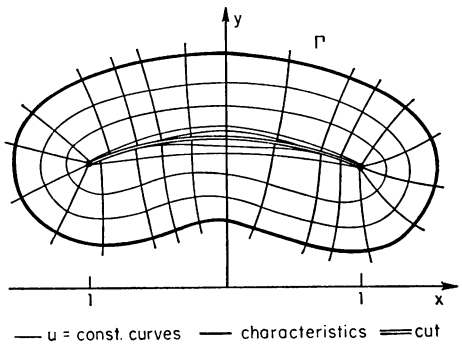

FIGURE 3

6. Cut analysis. The scaling factors h_1 and h_2, are well-behaved in a neighbourhood of the cut, except possibly at foci which are considered in the next section. The problem associated with the cut is therefore simply the problem of patching solutions across the cut and is akin to the "patching" problem encountered in the one-dimensional case of Part I. In Fig. 4, the two characteristics, C and D, meet the cut E, at F. Suppose the solution along C is

$$\psi_1 = AAi(\zeta_1) + \epsilon^2 BAi'(\zeta_1),$$

and the solution along D is

$$\psi_2 = \bar{A}Ai(\zeta_2) + \epsilon^2 \bar{B}Ai'(\zeta_2).$$

From the construction of the co-ordinate system, $\zeta_1 = \zeta_2 = \bar{\zeta}$ (say) at F. Now $\psi(\mathbf{x})$, being the solution of an elliptic partial differential equation, is required to have continuous derivatives of all order throughout the solution domain, and in particular along the cut. A necessary condition for this to be true is that ψ, and its first order derivatives, be continuous at every point F, of the cut E. If $\hat{n}$ and $\hat{s}$ are unit vectors perpendicular to,

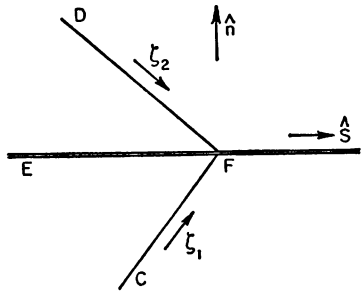

FIGURE 4

and tangential to the cut respectively, then these continuity conditions at F require;

$$\psi_1(F) = \psi_2(F),$$

when

$$\psi_{1s}(F) = \psi_{2s}(F), \tag{6.1}$$

and

$$\psi_{1n}(F) = \psi_{2n}(F);$$

i.e. to first order in ϵ,

$$\begin{aligned} AAi(\bar{\zeta}) &= \bar{A}Ai(\bar{\zeta}), \\ AAi'(\bar{\zeta})\cos(n\zeta_1) &= \bar{A}Ai'(\bar{\zeta})\cos(n\zeta_2), \\ AAi'(\bar{\zeta})\cos(s\zeta_1) &= \bar{A}Ai'(\bar{\zeta})\cos(s\zeta_2), \end{aligned} \tag{6.2}$$

where $(n\zeta_1)$ denotes the angle between $\hat{n}$ and the direction of increasing ζ_1 etc. (Use is made here of the fact that ∇u is continuous across the cut which follows from Eq. (3.3) and the fact that u is continuous across the cut.)

These relationships are to be valid for all points F of E. This requires either:

(I) $A = \bar{A} = O(\epsilon)$

or

(II) the cut, E, is perpendicular to both families of characteristics—Eqs. (6.2) are then consistent provided the A, $\bar{A}$ coefficient determinant, $2Ai(\bar{\zeta})\,Ai'(\bar{\zeta})$, vanishes or

(III) E is a limit line of the characteristics above and below E. No further consistency conditions are required in this case. A cut of this type has not been encountered in the examples considered, and it appears unlikely that such a double limit line will arise in practice. This case will not, therefore, be pursued further.

Case I. Because S is finite at the cut, A and $\bar{A}$ can only vanish to order ϵ at the cut, as a result of $F(\beta)$ and $\bar{F}(\beta)$ vanishing to order ϵ. (This follows from (4.9) and (4.10).) This implies that $A = \bar{A} = O(\epsilon)$ along the characteristics emanating from the cut, which in turn implies that $A = \bar{A} = B = \bar{B} = 0$, to terms of order ϵ^n for all n. Thus the solution $\psi(\mathbf{x})$, along characteristics emanating from a cut of this type, is zero to terms of order ϵ^n (a solution which is exponentially small in ϵ may exist in this "cut region"). Thus, under no circumstances can a standing wave pattern be set up in *the "cut region"* which *forms*, therefore, *a region of complete interference* in the solution domain. The situation envisaged is this. If the initial value problem was to be considered and disturbances were initially set up in the solution domain, then these disturbances would propagate back and forth along the characteristics, interfering with one another, finally leading to a steady state—the state being sought here. In the present case, the transition surface, which acts as a reflexion surface for the propagating waves, is such that no standing wave pattern can be set up in the "cut region", and the steady state solution vanishes due to annulment.

Case II. A nonzero solution exists in this case, provided one of the eigenconditions given to first order by $Ai(\bar{\zeta}) = O(\epsilon)$, or $Ai'(\bar{\zeta}) = O(\epsilon)$, is satisfied.

Of all the wave patterns that may be set up initially in this "cut region" (which will

be referred to as a *reflexion region* in the work to follow) only those which suffer in-phase reflexion at the transition surface will survive and will be present in the steady state. All others will interfere and annul one another. The eigenconditions above may be interpreted as being simply phase conditions for these steady states. If these conditions are satisfied, then the matching relations (6.1) are consistent, and may be solved. As in the one-dimensional case, these relations ensure the continuity of all higher order derivatives along the characteristics (i.e. in the direction of ∇u). The solution, $\psi(\mathbf{x})$, is also required to have continuous derivatives of all order in the β direction (i.e. in the u = constant direction).[6] To ensure this, the $F_k(\beta)$ for all $k \geq 0$, which occur in the coefficient terms of $Ai(\zeta)$ in the solution expansion (see Eq. (4.9)), must be of class ϵC^∞.[7] At this stage it might be noted that a focus may be thought of as a degenerate case of this type of cut—the case in which the cut degenerates to a point. It might be expected therefore, that the focus will lead to eigenstates in much the same way.

7. Focus analysis. An infinite band of characteristics intersect at a focus which therefore maps onto a line (parallel to the β axis) on the (α, β) plane. Thus at a focus the scaling factor S has a first order zero and $A_0(\alpha, \beta)(=F_0(\beta)/S^{1/2})$ is singular. Higher order terms in the expansions for A and B (see (4.9), (4.10)) are still more singular there. The solution $\psi(\mathbf{x})$ is required to be well behaved at the focus so the expansion (4.12) does not truly represent the solution behaviour close to this point. An investigation of the solution behaviour close to such a mapping singularity is now undertaken.

Close to the focus, α_0 say, the scaling factor S may be usefully represented by a series expansion of the form

$$S = s_0(\beta)(\xi + s_1(\beta)\xi^2 + \cdots) \tag{7.1}$$

where

$$\xi = (\alpha - \alpha_0), \tag{7.2}$$

and (because the scaling factor is, by definition, always ≥ 0) $s_0(\beta) > 0$. [In the centrally symmetric case ($g(\mathbf{x}) = g(r)$) $s_0(\beta)$ reduces to a constant]. Similarly since α is well behaved in the neighbourhood of the focus a power series expansion of the form

$$\alpha = \alpha_0 + u_1(\beta)\xi + \cdots, \tag{7.3}$$

is assumed. The value of α at the focus is determined by the scale of $g(\mathbf{x})$, and a change in the scale of g can be effected by simply altering the value of ϵ. Thus there is no loss in generality in assuming that the value of α at the focus is unity (i.e. $\alpha_0 = 1$).[8]

The expansion (4.12) for $\psi(\mathbf{x})$ is invalid close to the focus *only* because the coefficient expansions (4.1) are singular there. One way of dealing with this difficulty would be simply to modify the expansions (4.1) for A and B in such a way that they coped with the situation close to the focus. This could be done, for example, by seeking expressions for A and B of the form $A = A(\mathbf{x}, \omega, \epsilon)$ and $B = B(\mathbf{x}, \omega, \epsilon)$, where ω is a suitable chosen locally stretched co-ordinate, introduced to cope with the singular character of the Eqs. (3.4) and (3.5) for A and B close to the focus, α_0 . If a uniformly valid asymptotic expansion is to be sought, this is clearly an obvious way to go about it, because it is

[6]For the equation which $\psi(\mathbf{x})$ satisfies is elliptic.

[7]Later (see Sec. 8) the further restrictions that need to be imposed on $F_k(\beta)$ will become evident.

[8]A reduction in the amount of algebra is achieved by doing this for the present general discussion, but may not be desirable in a particular problem.

essential to retain the $Ai(\zeta)$ dependence in $\psi(\mathbf{x})$ in order to cope with the situation close to Γ. However, it will be remembered that in the two turning points ordinary differential equation problem of Part I a similar situation arose. There, in order to obtain a simpler description of the solution behaviour, the requirement that the solution be described in terms of a *single* expansion was relaxed, and two expansions were used to describe the solution behaviour. In this case the Eqs. (3.4) and (3.5) determining A and B, are quite complex, and in order to obtain equations which are at all tractable it is highly desirable, if not necessary, to relax the single representation requirement, and to seek a separate description (the "inner expansion") of the solution $\psi(\mathbf{x})$, valid close to the focus.

The Inner Expansion. Equation (1.1), when referred to the (α, β) co-ordinate system, reduces to

$$\epsilon^3 \frac{1}{S}\left[\frac{\partial}{\partial\alpha}\left(S\frac{\partial\psi}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{S}\frac{\partial\psi}{\partial\beta}\right)\right] - u\psi = 0. \tag{7.4}$$

(The relationship $h_1 = u/g$ which follows from the scaling factor definition (4.8) and the definition (3.3) for u, is used here.)

Consider a δ neighbourhood of the focus α_0. Within this neighbourhood, S is of order δ, and u is of unit order. Thus a balance is achieved between the various terms of Eq. (7.4) if δ is of order $\epsilon^{3/2}$. This suggests that Eq. (7.4) might be better described close to α_0 in terms of the stretched co-ordinate $\tau = (\alpha_0 - \alpha)/\delta$, where $\delta = \epsilon^{3/2}$. In terms of this stretched co-ordinate the equation reduces to

$$\frac{1}{\bar{s}}\left[\frac{\partial}{\partial\tau}\left(\bar{s}\frac{\partial\psi}{\partial\tau}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{\bar{s}}\frac{\partial\psi}{\partial\beta}\right)\right] - u\psi = 0, \tag{7.5}$$

where

$$\bar{s} = S/\delta \tag{7.6}$$

and the expansions (7.3) and (7.1), for α and S, become

$$\alpha = 1 + \delta u_1(\beta)\tau + \cdots, \tag{7.7}$$

and

$$\bar{s} = \tau s_0[1 + \delta s_1\tau + \cdots]. \tag{7.8}$$

On substituting the expansions (7.7) and (7.8) into (7.5), and neglecting terms of order δ, the following approximate equation for ψ is obtained;

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\psi}{\partial\tau}\right) + \frac{1}{\tau^2}\frac{1}{s_0}\frac{\partial}{\partial\beta}\left(\frac{1}{s_0}\frac{\partial\psi}{\partial\beta}\right) + \psi = 0. \tag{7.9}$$

It will be remembered (see Sec. (4)) that the choice of β was left open. Equation (7.9) indicates that a most convenient choice of β is that defined in such a way that $s_0(\beta) = 1$ at the focus. A β so defined, simply measures the angular displacement of characteristics at the focus. This particular choice will be employed in the work to follow. Equation (7.9) is separable, and the solutions which are bounded at $\tau = 0$ are of the form

$$\psi_i = K(\delta, \epsilon)f(\gamma\beta)J_\gamma(\tau) \tag{7.10}$$

where $f(\gamma\beta)$ denotes

$$c\sin(\gamma\beta + d) \tag{7.11}$$

where c and d are arbitrary constants of integration, and γ is the separating constant whose values are thrown up later when appropriate boundary conditions are imposed on the solution $\psi(\mathbf{x})$. It will be seen that the outer solution (represented by (4.12)) on reaching the focus is not of unit order in ϵ, so that it is necessary to include the scaling factor $K(\delta, \epsilon)$ in the above expression for ψ_i .

Matching. The required solution, $\psi(\mathbf{x})$, is represented by the "inner expansion" whose first term is ψ_i (given by (7.10)) in the neighbourhood of the focus α_0 , and by the outer expansion (4.12), away from this point. The Lagerstrom, Kaplun *matching by rearrangement* technique is now employed to connect up these two expansions, and so evaluate the unknown quantities at present associated with each.

For large $+ve\tau$,

$$\psi_i \sim (2/\pi)^{1/2} K(\delta, \epsilon) f(\gamma\beta)/\tau^{1/2} \cos(\tau - (\gamma\pi/2) - (\pi/4)), \tag{7.12}$$

where use has been made of the asymptotic formulae for the Bessel functions.

Because ζ is of order $\epsilon^{-3/2}$ at α_0 , $Ai(\zeta)$ can be approximated by its asymptotic form i.e.

$$Ai(\zeta) \sim \frac{1}{\pi^{1/2}} \frac{1}{\zeta^{1/4}} \left\{ \cos\left(\frac{2}{3}\zeta^{3/2} - \frac{\pi}{2}\right) \right\},$$

with the result that the outer expansion (4.12) close to α_0 may be approximated by

$$\psi \sim \frac{1}{\pi^{1/2}} \frac{F_0(\beta)}{\tau^{1/2}} \left(\frac{\epsilon^{1/4}}{\delta^{1/2}}\right) \cos\left(\tau + \frac{\pi}{4} - \frac{2}{3}\frac{\alpha_0^{3/2}}{\epsilon^{3/2}}\right). \tag{7.13}$$

where use is made of the stretched co-ordinate relation

$$\tau = (\alpha_0 - \alpha)/\delta.$$

Assuming that there is a nonzero domain in which both (7.12) and (7.13) are valid representations of the solutions $\psi(\mathbf{x})$, the following matching relations are obtained:

$$K(\delta, \epsilon) = (\epsilon^{1/4}\ \delta^{-1/2}), \qquad F_0(\beta) = f(\gamma\beta)/2^{1/2}, \tag{7.14}$$

and

$$\frac{2}{3}\frac{1}{\epsilon^{3/2}} = \pm\left(\frac{\gamma\pi}{2} - \frac{\pi}{4}\right) + \frac{\pi}{4} + n\pi, \tag{7.15}$$

where n is any large positive integer. Thus, the solution along characteristics leading into a focus (except for a small neighbourhood of that point) to first order is given by

$$\psi = \frac{f(\gamma\beta)}{(2S)^{1/2}} Ai(\zeta), \tag{7.16}$$

and the standing wave patterns represented by this expression can only be set up if the radial "phase" eigencondition (7.15) (which determines eigenvalues for ϵ) is satisfied to order ϵ. It is more convenient to write this eigencondition in the form;

$$\zeta(\alpha_0) = \zeta_s \pm \gamma\pi/2,$$

where ζ_s is determined from either

$$Ai(\zeta_s) = 0, \tag{7.17}$$

or

$$Ai'(\zeta_s) = 0.$$

The azimuthal eigencondition is determined when appropriate azimuthal boundary conditions are imposed or $\psi(\mathbf{x})$. If the focus is isolated the solution $\psi(\mathbf{x})$ is single-valued only if γ takes on zero or positive integer values. Thus along the characteristics emanating from an *isolated focus* the *solution* is given by

$$\psi = \frac{c \sin(\gamma\beta + \alpha)}{S^{1/2}} Ai(\zeta)[1 + O(\epsilon^{3/2})]$$

where

$$\gamma = 0, 1, 2, \cdots \tag{7.18}$$

and

$$\zeta(\alpha_0) = \zeta_s \pm \gamma\pi/2$$

where either $Ai'(\zeta_s) = 0$ or $Ai(\zeta_s) = 0$. If the focus is not isolated (i.e. it lies on a cut[9]) then the boundary conditions are obtained from the solution differentiability requirements across the characteristics separating the focus domain and the "reflection" or "interference" (see Sec. 6) cut domains. In order to realize these requirements, it is necessary to postulate the existence of two sided boundary layers (see Fig. 5) in the neighbourhood of the "separating characteristics", the presence of which can be inferred from the equations for A and B.

The boundary layer equations can be readily integrated and it can be shown that the required differentiability conditions can then be satisfied if,

$$f(\gamma\beta_0) = 0 + O(\epsilon^{3/2})$$

a detailed derivation of this result is given in [2].

This boundary layer behaviour of the solution in the neighbourhood of the separating characteristics may be thought of as a dispersion phenomenon. Thus to first order the solution in the *focus domain bounded* by the *separating* characteristics β_1 and β_2 is given by,

$$\psi = \frac{\text{const} \sin[\gamma(\beta - \beta_1)]}{S^{1/2}} Ai(\zeta)[1 + O(\epsilon^{3/2})]$$

when

$$\gamma = 2\pi n/(\beta_2 - \beta_1) \qquad \text{for } n = 1, 2, \text{ etc.} \tag{7.19}$$

and as before ϵ satisfies,

$$\zeta(\alpha_0) = \zeta_s \pm \gamma\pi/2$$

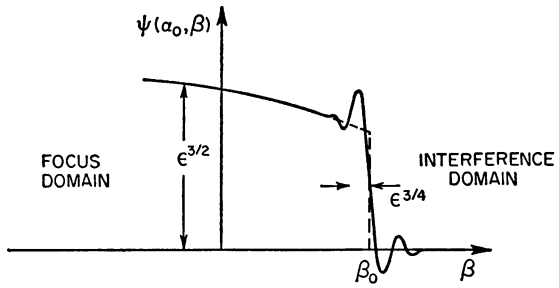

FIGURE 5

[9]Which for example occurs in the situations depicted in Figs. 1 and 2 and in the two foci example of Fig. 3.

In the two foci problem (see Fig. 3) the above work indicates that, for the modes we are considering, the solution is exponentially small in ϵ everywhere except in the two end foci domains.

NOTE. It is *not necessary* to *undertake any of the laborious analysis* associated with the focus, matching, and the separating characteristics *in order to obtain the first order solution behaviour.*[10]

8. Discussion and summary. Earlier, in order to be explicit, a particular procedure for uniquely covering the (x, y) plane with an (α, β) co-ordinate net (see section (5)) was prescribed. There it was surmised, on the basis of experience gained in the one-dimensional case of Part I, that all cut configurations would give equivalent results. A complete discussion of this point is perhaps not warranted here,[11] and therefore, it will suffice to indicate by way of a simple example the qualifications to the statement that need to be made. For this purpose the infinite boundary wall, constant depth rectangular potential well defined by

$$g(x, y) = -1 \qquad \text{for} \quad |x| \leq a \quad \text{and} \quad |y| \leq b,$$

$$g(x, y) = \infty \qquad \text{for} \quad |x| > a \quad \text{and} \quad |y| > b$$

is considered (see Fig. 6). We know on physical grounds what to expect in this case. Since the boundaries act as reflecting walls, modes can be set up by reflection between any two opposite boundary segments.

Now the cutting procedure prescribed earlier leads to cuts running from E and F to all four corners, as shown in Fig. 6a. This configuration displays the "dashed" reflection domain shown in this figure. The $E'F'$ cut configuration seen on the same figure can be shown to indicate the presence of the same modes and therefore is for our purpose equivalent. In fact all the obvious seemingly equivalent configurations can be shown to be equivalent. The cut configuration displayed in Fig. 6b indicates the presence of a class of "horizontal" reflection modes missed by our prescribed cutting procedure. All the other modes suggested by the physics of the problem can obviously be displayed by other cut configurations. It is clear from this that in complicated situations it may be necessary to sketch the complete diagram of characterictics in order to extract all possible modes. Usually, however, the problem will be best attacked by consideration of the physics or geometry appropriate to the given problem.

Conclusion. It should be remembered that only those eigenmodes that vary slowly around the transition surface have been extracted by the preceding work. These eigenmodes have been obtained for quite general $g(x, y)$. To do this it was found necessary *only* to solve the non-linear partial differential equation

$$u(\nabla u)^2 = g(\mathbf{x}),$$

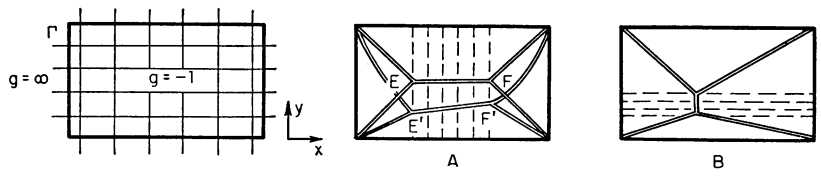

FIGURE 6

[10]The solution as given by (7.18) is not dependent on the detailed solution behaviour at the focus etc.

[11]A more complete discussion is given in [2].

with $u = 0$ around the transition surface, using the method of characteristics—a straightforward calculation in a digital computer. Either by consideration of the physics of the problem or by use of a complete characteristic diagram the foci and reflection regions can be picked out. The solution along characteristics leading into foci is then immediately given by (7.18) or (7.19). In the reflection regions the solution is given by (4.11) when the eigencondition requirement,

$$Ai(\bar{\zeta}) = O(\epsilon) \quad \text{or} \quad Ai'(\bar{\zeta}) = O(\epsilon)$$

is satisfied. In all other regions of the solution domain the solution vanishes to order ϵ^N for all N. Regions that are not crossed by characteristics are simply shadow regions in the solution domain.

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