ON VELOCITY CORRELATIONS AND THE SOLUTIONS OF THE EQUATIONS OF TURBULENT FLUCTUATION*

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1. Introduction. The theory of turbulence, as developed from Reynolds' point of view, is based upon the equations of turbulent fluctuation [1] and has been applied to the solutions of various special problems [2, 3, 4, 5, 6, 7]. Owing to present circumstances, these papers either have not been submitted to scientific journals for publication or are already printed but have failed to appear before the scientific public. The theory in its original form and its applications has three apparent difficulties: first, the equations of correlation of the second, third or even higher orders constructed out of the equations of turbulent fluctuation contain the unknown terms of correlation between the pressure and velocity fluctuations; secondly, there exist in these equations the terms of decay of turbulence the values of which have to be determined; thirdly, when the differential equations of the velocity correlations of a given order are derived from the equations of turbulent fluctuation, the presence of the inertia terms causes the appearance of the velocity correlations of the next higher order, which are also unknown. This has been pointed out by von Kármán and Howarth [8] in their theory of homogeneous isotropic turbulence.

In the present paper we shall show that the pressure fluctuation can be derived from the equations of turbulent fluctuation, and is expressible as a function of the velocity fluctuation, the mean velocity inside the fluid volume, and the pressure fluctuation on the boundary. We shall also show that the decay terms can be put into simpler and more familiar forms by kinematic considerations. A general equation of vorticity decay will be derived for the determination of Taylor's scale of the micro-turbulence which appears in the decay term; in the case of homogeneous isotropic turbulence, this equation was given first by von Kármán [8]. To get over the third difficulty we shall compare the orders of magnitudes of the different terms in the equations of triple correlation. We shall find that the term involving the divergence of the quadruple correlation is actually smaller than the correlation between the pressure gradient and the two components of velocity fluctuation, and can therefore be neglected as a first approximation. From this we can also understand why, for the flows in channels and pipes in which the mean velocity profile is comparatively steep, particularly in the neighborhood of the walls, all the equations of mean motion and the equations of double and triple correlation are necessary to describe the phenomena of turbulent motions of fluids. On the other hand, as a consequence of the approximation based on the fact that the divergence of the quadruple correlation is smaller than the correlation between the pressure gradient and the two components of velocity fluctuation, we can stop at the equations of triple correlation instead of building equations of higher orders. As a matter of fact, for the flows in jets [3] and wakes [4]

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where no wall is present, the equations of mean motion and of double correlation are sufficient, after some simple approximations to the triple correlations are made, for the determination of the mean velocity distribution, and the equations of triple correlation can be dispensed with.

From a mathematical point of view the present program indicates that the turbulence problem can be reduced rigorously to a set of non-linear partial integro-differential equations the solutions of which are very difficult to ascertain. In order to facilitate the solution of special problems, approximate forms of the integral parts of the equations have been developed in a general way. These approximations, however, are only valid in regions not too close to the boundary of the moving fluid volume. It may also be worthwhile to point out that the unsatisfactory part of the present theory lies in the uncertain nature of the correlation integrals, as will be seen presently in §8. A better and more accurate representation of these integrals is possible, provided more accurate experimental information can be obtained as to the distribution of turbulence levels and to the correlation functions between two distinct points in general.

The rigorous way of treating the turbulence problem is probably to solve the Reynolds' equations of mean motion and the equations of turbulent fluctuation simultaneously. This procedure, however, is very difficult owing to the non-linearity of the two sets of equations. Hence we have adopted the method of solving the equations of turbulent fluctuation by setting up the differential equations satisfied by the velocity correlation functions of different orders, a method initiated by von Kármán and Howarth [8] in treating the problem of homogeneous isotropic turbulence. This process of setting up the correlation equations of different orders and seeking their solution can be regarded as a method of successive approximation to the solution of the turbulence problem; it will be explained in the concluding section of the present paper. The correlation functions of higher orders in the various special problems, obtained by this setting-up process, should be verifiable by direct observation with the advance of modern experimental technique; at present experiments have only been performed to measure the mean velocity distribution and the second order stress tensors in a turbulent flow. It should also be noted that although the equations of correlation have a much more complicated mathematical appearance than that of the Navier-Stokes' differential equations from which they are derived, the method of Prandtl's boundary layer approximations can still be used without leading to contradictions for the particular problems [3, 4, 5] under consideration.

For the sake of convenience we list below the different equations of motion which have been derived heretofore [1]. Reynolds' equations of mean motion and the equation of continuity for an incompressible fluid are given by

$$\frac{\partial U_i}{\partial t} + U^j U_{i,j} = -\frac{1}{\rho}\bar{p}_{,i} + \frac{1}{\rho}\tau^j{}_{i,j} + \nu\nabla^2 U_i, \qquad U^j{}_{,j} = 0. \tag{1.1}$$

Here, the tensor notation is employed, and U_i are the velocity components of the mean motion, t is the time, ρ is the density, $\bar{p}$ is the mean pressure, ν is the coefficient of kinematic viscosity, a subscript preceded by a comma denotes the covariant derivative, ∇^2 denotes the Laplacian operator, and Reynolds' apparent stress $\tau^j{}_i$ is defined by the relation

$$\tau^i_{\ i} = -\overline{\rho w_i w^j}, \tag{1.2}$$

w_i being the velocity components of the turbulent motion.

The equations of turbulent fluctuation and the equation of continuity for the velocity fluctuation w^i, which are the differences of the Navier-Stokes' equations and Reynolds' equations (1.1), are

$$\frac{\partial w_i}{\partial t} + U^j w_{i,j} + w^j w_{i,j} + w^j U_{i,j} = -\frac{1}{\rho}\varpi_{,i} - \frac{1}{\rho}\tau^j_{\ i,j} + \nu\nabla^2 w_i, \qquad w^j_{\ ,j} = 0, \tag{1.3}$$

where ϖ is the pressure fluctuation. From the above set of equations we derive the equations of vorticity fluctuation,

$$\frac{\partial}{\partial t}\omega_{ik} + U^j\omega_{ik,j} + U^j_{\ ,k}w_{i,j} - U^j_{\ ,i}w_{k,j} + w^j\omega_{ik,j} + w^j_{\ ,k}w_{i,j} - w^j_{\ ,i}w_{k,j}$$
$$+ w^j\Omega_{ik,j} + w^j_{\ ,k}U_{i,j} - w^j_{\ ,i}U_{k,j} = -\frac{1}{\rho}(\tau^j_{\ i,jk} - \tau^j_{\ k,ji}) + \nu\nabla^2\omega_{ik}, \tag{1.4}$$

where the mean vorticity Ω_{ik} and the vorticity fluctuation ω_{ik} are defined by the equations

$$\Omega_{ik} = U_{i,k} - U_{k,i}, \qquad \omega_{ik} = w_{i,k} - w_{k,i}. \tag{1.5}$$

The equations of double velocity correlation derived from (1.3) are

$$-\frac{1}{\rho}\frac{\partial\tau_{ik}}{\partial t} - \frac{1}{\rho}(U_{i,j}\tau^j_{\ k} + U_{k,j}\tau^j_{\ i}) - \frac{1}{\rho}U^j\tau_{ik,j} + (\overline{w^j w_i w_k})_{,j}$$
$$= -\frac{1}{\rho}(\overline{\varpi_{,i}w_k} + \overline{\varpi_{,k}w_i}) - \frac{\nu}{\rho}\nabla^2\tau_{ik} - 2\nu g^{mn}\overline{w_{i,m}w_{k,n}}, \tag{1.6}$$

where the superimposed bar denotes the mean. The ten equations of triple correlation are

$$\frac{\partial}{\partial t}\overline{w_i w_k w_l} + U_{i,j}\overline{w^j w_k w_l} + U_{k,j}\overline{w^j w_l w_i} + U_{l,j}\overline{w^j w_i w_k} + U^j(\overline{w_i w_k w_l})_{,j} + (\overline{w^j w_i w_k w_l})_{,j}$$
$$= -\frac{1}{\rho}(\overline{\varpi_{,i}w_k w_l} + \overline{\varpi_{,k}w_l w_i} + \overline{\varpi_{,l}w_i w_k})$$
$$+ \frac{1}{\rho^2}(\tau^j_{\ i,j}\tau_{kl} + \tau^j_{\ k,j}\tau_{li} + \tau^j_{\ l,j}\tau_{ik}) + \nu g^{mn}(\overline{w_i w_k w_l})_{,mn}$$
$$- 2\nu g^{mn}(\overline{w_{i,m}w_{k,n}w_l} + \overline{w_{k,m}w_{l,n}w_i} + \overline{w_{l,m}w_{i,n}w_k}). \tag{1.7}$$

2. The pressure fluctuation. Let us take the divergence of the equations of turbulent fluctuation (1.3). Because of the equation of continuity satisfied by w^i, the pressure fluctuation ϖ satisfies the following Poisson's equation:

$$\frac{1}{\rho}\nabla^2\varpi = -2U^m_{\ ,n}w^n_{\ ,m} + (\overline{w^m w^n} - w^m w^n)_{,mn}. \tag{2.1}$$

Since any two successive covariant differentiations are commutative in a Euclidean space, the gradient of the pressure fluctuation $\tilde{\omega}_{,k}$ also satisfies a Poisson's equation,

$$\frac{1}{\rho}\nabla^2\varpi_{,k} = -2(U^m{}_{,n}w^n{}_{,m})_{,k} + (\overline{w^m w^n} - w^m w^n)_{,mnk}. \tag{2.2}$$

The general solution of (2.2) can be written in the form,

$$\frac{1}{\rho}\varpi_{,k} = \frac{1}{2\pi}\iiint (U'^m{}_{,n}w'^n{}_{,m})'_{,k}\frac{1}{r}dV' - \frac{1}{4\pi}\iiint (\overline{w'^m w'^n} - w'^m w'^n)'_{,mnk}\frac{1}{r}dV'$$
$$+ \frac{1}{4\pi\rho}\iint\left\{\frac{1}{r}\frac{\partial\varpi'_{,k}}{\partial n'} - \varpi'_{,k}\frac{\partial}{\partial n'}\left(\frac{1}{r}\right)\right\}dS', \tag{2.3}$$

where the integrations extend over the whole region of the moving fluid, the first two integrals represent the particular integrals, and the third represents the complementary solution which is a harmonic function expressed in terms of the boundary values of itself and its normal derivative; x'^i are the coordinates of a point P' which ranges over the region of the moving fluid, r is the distance from P' to the point P with coordinates x^i, dV' is a volume element, dS' is a surface element, $\partial/\partial n'$ denotes the normal derivative, and the primes on the various quantities on the right side of (2.3) indicate that these quantities are to be evaluated at P'. We shall see finally that the surface integral in (2.3) can be neglected for points P where $\varpi_{,k}$ is defined and which are not too close to the boundary of the moving fluid.

We now let both x^i and x'^i represent rectangular cartesian coordinates, and let ξ^i denote the difference vector of x'^i and x^i, i.e.,

$$\xi^i = x'^i - x^i. \tag{2.4}$$

Covariant differentiation then reduces to ordinary differentiation, and the difference between covariant and contravariant tensor character disappears. Hence ξ^i is equal to ξ_i and the distance r between P and P' is given by

$$r^2 = \xi_i\xi^i. \tag{2.5}$$

The element of volume dV' is equal to $d\xi^1 d\xi^2 d\xi^3$.

The solution (2.3) clearly shows that besides the harmonic function expressed as a surface integral on the boundary, the pressure fluctuation at a point P, and its gradient, are determined by the turbulent velocity fluctuation w^i not only at P but also everywhere within the fluid. However, due to the factor $1/r$ in the integrands the effect of the velocity fluctuation at distant points P' on the pressure fluctuation at P gradually dies away as P' recedes farther and farther from P.

3. Velocity correlation between two distinct points. The partial differentiations in the integrand functions in (2.3) are taken with respect to the coordinates x'^k which are independent of x^i. Hence, if we multiply (2.3) by the velocity fluctuation w_i at the point P, we obtain the correlation between w_i and $\varpi_{,k}$ at the same point P:

$$\frac{1}{\rho}\overline{\varpi_{,k}w_i} = \frac{1}{2\pi}\iiint [U'^m{}_{,n}(\overline{w'^n w_i})'_{,m}]'_{,k}\frac{1}{r}dV' + \frac{1}{4\pi}\iiint (\overline{w'^m w'^n w_i})'_{,mnk}\frac{1}{r}dV'$$
$$+ \frac{1}{4\pi\rho}\iint\left\{\frac{1}{r}\frac{1}{\partial n'}(\overline{\varpi'_{,k}w_i}) - \overline{\varpi'_{,k}w_i}\frac{\partial}{\partial n'}\left(\frac{1}{r}\right)\right\}dS'. \tag{3.1}$$

We shall neglect, however, the surface integral in the above equation on the ground that the correlation $\overline{\varpi'_{,k}w_i}$ is small provided that the point P where the correlation $\overline{\varpi_{,k}w_i}$ is under consideration is situated not too close to the boundary. This condition limits the present theory to regions where free turbulence predominates.

Likewise, under the same condition of approximation the correlation function $\rho^{-1}\overline{\varpi_{,k}w_l w_i}$ is given by

$$\frac{1}{\rho}\overline{\varpi_{,k}w_l w_i} = \frac{1}{2\pi}\iiint [U'^m{}_{,n}(\overline{w'^n w_l w_i})'_{,m}]'_{,k}\frac{1}{r}dV'$$
$$-\frac{1}{4\pi}\iiint [\overline{w'^m w'^n}\,\overline{w_l w_i} - \overline{w'^m w'^n w_l w_i}]'_{,mnk}\frac{1}{r}dV'. \qquad (3.2)$$

If we solve for ϖ from (2.1) and form its correlation with w_i, we find, to the same order of approximation, that

$$\frac{1}{\rho}\overline{\varpi w_i} = \frac{1}{2\pi}\iiint U'^m{}_{,n}(\overline{w'^n w_i})'_{,m}\frac{1}{r}dV' + \frac{1}{4\pi}\iiint (\overline{w'^m w'^n w_i})'_{,mn}\frac{1}{r}dV'. \qquad (3.3)$$

In the three equations (3.1), (3.2) and (3.3) we recognize three types of functions, namely, $\overline{w'^m w_i}$, $\overline{w'^m w'^n w_i}$ and $\overline{w'^n w_l w_i}$, and $\overline{w'^m w'^n w_l w_i}$; they are, according to Taylor [10] and von Kármán [8], the velocity correlations between two distinct points P and P' of the second, third and fourth orders respectively. They are usually functions of both the coordinates x^i and x'^k and probably also of the time t. The double correlation function $\overline{w'^n w_i}$ between P and P' has been measured extensively for isotropic turbulence by several authors [11, 12]; for flow in a channel [13] and in a pipe [14], they have been recorded only in a number of isolated cases and only within limits.

It has been observed that for isotropic turbulence $\overline{w'^n w_i}$ vanishes very rapidly for large values of the quantities ξ^i defined in (2.4). This must also hold true for the other two correlation functions $\overline{w'^m w'^n w_i}$ and $\overline{w'^n w_l w_i}$, and also for other types of flow; furthermore their derivatives with respect to ξ^k should all approach zero rapidly with increasing ξ^i.

On the other hand, the quadruple correlation $\overline{w'^m w'^n w_l w_i}$ between the points P and P' does not necessarily vanish when P and P' are widely separated, for the average values of both $\overline{w'^m w'^n}$ and $\overline{w_l w_i}$ over a period of time τ are themselves not separately equal to zero in general. Hence as an analogy to the velocity vector W^i, we may separate the product $w_l w_i$ into two parts, the correlation $\overline{w_l w_i}$ and a symmetric tensor u_{li} the time average of which vanishes,*

$$w_l w_i = \overline{w_l w_i} + u_{li}, \qquad (3.4)$$
$$\overline{u_{li}} = \frac{1}{\tau}\int_{\tau-\tau/2}^{\tau+\tau/2} u_{li}dt = 0.$$

An analogous relation holds good for $\overline{w^m w'^n}$. The quadruple correlation $\overline{w'^m w'^n w_l w_i}$ between the points P and P' consequently becomes

$$\overline{w'^m w'^n w_l w_i} = \overline{w'^m w'^n}\,\overline{w_l w_i} + \overline{u'^{mn} u_{li}}. \qquad (3.5)$$

As P and P' recede farther and farther from each other, the correlation function

* The author wishes to express his gratitude to Mr. S. L. Chang for pointing out relation (3.4).

$\overline{u'^{mn}u_{li}}$, which behaves like $\overline{w'^n w_i}$, will tend toward zero as a limit. Substitution of (3.5) into (3.2) yields

$$\frac{1}{\rho}\overline{\varpi_{,k}w_l w_i} = \frac{1}{2\pi}\iiint [U'^m{}_{,n}(\overline{w'^n w_l w_i})'_{,m}]'_{,k}\frac{1}{r}dV' + \frac{1}{4\pi}\iiint (\overline{u'^{mn}u_{li}})'_{,mnk}\frac{1}{r}dV'. \tag{3.6}$$

If we substitute in the equations of the double and triple correlations (1.6) and (1.7) for $\rho^{-1}\overline{\varpi_{,k}w_i}$ from (3.1) and $\rho^{-1}\overline{\varpi_{,k}w_l w_i}$ from (3.6) above, we obtain a set of integro-differential equations for the mean velocity, the double and triple velocity correlations of a turbulent flow at a point P being the dependent variables with the velocity correlations between two distinct points $\overline{w'^n w_i}$ and $\overline{w'^n w_l w_i}$ as kernels. This set of integro-differential equations is too complicated for solving special problems, so we shall presently develop approximate forms of the integral parts of the equations in a general way.

It should be noted that for homogeneous isotropic turbulence the following relation between the triple correlations holds [8]:

$$\overline{w'_m w'_n w_i} = -\overline{w'_i w_m w_n}. \tag{3.7}$$

4. Conservation relations satisfied by the velocity correlations. The velocity fluctuation w'^n at the point P' satisfies the equation of continuity $w'^n{}_{,n}=0$. Let us multiply this equation by w_i and average over an interval of time τ. Since P and P' are independent, we obtain the conservation equation for the double correlation $\overline{w'^n w_i}$ between P and P',

$$\frac{\partial}{\partial x'^n}(\overline{w'^n w_i})_x = 0, \tag{4.1}$$

where the coordinates are still rectangular cartesian, and the subscript x indicates that the variables x^k are to be held constant while the differentiation is carried out.

Instead of x^i and x'^i, we can use the coordinates x^i and ξ^i, i.e., we transform from the old variables x^i and x'^i to the new variables x^i and ξ^i by means of the equations

$$x^i = x^i, \qquad x'^i = x^i + \xi^i. \tag{4.2}$$

In terms of the new coordinates x^i and ξ^i, Eq. (4.1) becomes

$$\frac{\partial}{\partial x'^n}(\overline{w'^n w_i})_x = \frac{\partial}{\partial \xi^n}(\overline{w'^n w_i})_x = 0. \tag{4.3}$$

For the sake of simplicity we shall drop the subscript x in (4.3); it will be understood that the variables x^k are regarded as constants during the differentiation. Hence we can write the divergence equation (4.3) in the covariant form,

$$(\overline{w'^n w_i})_{,n} = 0. \tag{4.4}$$

Similarly, from the equation of continuity for w^i at the point P, we have

$$\frac{\partial}{\partial x^i}(\overline{w'_n w^i})_{x'} = 0.$$

In terms of the new coordinates x^i and ξ^i, this relation becomes

$$\frac{\partial}{\partial x^i}\,(\overline{w'_n w^i})_\xi - \frac{\partial}{\partial \xi^i}\,(\overline{w'_n w^i})_x = 0.$$

After changing the variables from x^i, x'^i to x^i, ξ^i it can be seen that $\overline{w'^n w_i}$, considered as a function of x^i and ξ^i rather than of x^i and x'^i, varies slowly with x^i but rapidly with ξ^i for points not too close to the boundary of the fluid volume. Hence, as a first approximation the equation of conservation for the double correlation between P and P' in the index i is given by

$$(\overline{w'_n w^i})_{,i} = 0. \tag{4.5}$$

We note that to the first approximation the correlation function $\overline{w_k' w_i}$ satisfies the conservation equation symmetrically with respect to the indices i and k.

Likewise, the other two correlation functions $\overline{w'^m w'^n w_i}$ and $\overline{w'^n w_i w_k}$ between P and P' can be shown to satisfy the following relations:

$$(\overline{w'^m w'^n w^i})_{,i} = 0, \qquad (\overline{w'^n w_i w_k})_{,n} = 0. \tag{4.6}$$

The first equation in (4.6) is derived by an approximation as was (4.5); the second one is rigorous. We must not forget that all the covariant derivatives in (4.5) and (4.6) are taken with respect to the variables ξ^i, the coordinates x^i being held constant.

It is obvious that since the coordinates x^i of the point P are regarded as constants under the integrations in (3.1), (3.2) and (3.3), the covariant derivatives with respect to x'^k in the integrand functions can all be replaced rigorously by covariant derivatives with respect to the variables ξ^k, because of the equations of coordinate transformation (4.2). For example, (3.1) then becomes

$$\frac{1}{\rho}\,\overline{\varpi_{,k} w_i} = \frac{1}{2\pi}\iiint [U'^m{}_{,n}(\overline{w'^n w_i})_{,m}]_{,k}\,\frac{1}{r}\,dV'$$
$$+\frac{1}{4\pi}\iiint (\overline{w'^m w'^n w_i})_{,mnk}\,\frac{1}{r}\,dV'. \tag{4.7}$$

The other two integrals (3.3) and (3.6) can be altered analogously

5. Correlation integrals between the pressure gradient and velocity fluctuations. Let us examine the integral (4.7) more closely. In the integrand function of the first integral on the right hand side, $U'^m{}_{,n}$ is a more slowly varying function of ξ^i than its factor $\overline{w'^n w_i}$, both functions being regarded as functions of x^k and ξ^i. Hence, we expand $U'^m{}_{,n}$ at the point P' in a multiple power series in ξ^i,

$$\frac{\partial U'^m}{\partial x'^n} = \frac{\partial U^m}{\partial x^n} + \sum_{s=1}^{\infty}\frac{1}{s!}\,\frac{\partial^{s+1}U}{\partial x^{l_1}\partial x^{l_2}\cdots\partial x^{l_s}\partial x^n}\,\xi^{l_1}\xi^{l_2}\cdots\xi^{l_s}. \tag{5.1}$$

Substitution of (5.1) into (4.7) would yield a series of integrals which would be too complicated for any practical application. But if we neglect the higher order terms, in (5.1), then we have as a first approximation to (4.7),

$$\frac{1}{\rho}\,(\overline{\varpi_{,i} w_k} + \overline{\varpi_{,k} w_i}) = a^n{}_{mik} U^m{}_{,n} + b_{ik}\,. \tag{5.2}$$

where the functions $a^n{}_{mik}$ and b_{ik} are defined by

$$a^n{}_{mik} = \frac{1}{2\pi}\iiint [(\overline{w'^n w_i})_{,mk} + (\overline{w'^n w_k})_{,mi}] \frac{1}{r}\, dV',$$

$$b_{ik} = \frac{1}{4\pi}\iiint [(\overline{w'^m w'^n w_i})_{,mnk} + (\overline{w'^m w'^n w_k})_{,mni}] \frac{1}{r}\, dV'. \tag{5.3}$$

Owing to the conservation relations (4.5) and (4.6), the above two sets of functions also satisfy the following divergence conditions:

$$a^n{}_{nik} = 0, \qquad g^{ik} a^n{}_{mik} = 0, \qquad g^{ik} b_{ik} = 0. \tag{5.4}$$

Of these three conservation relations, the first follows from the rigorous continuity equation (4.4) and is hence exact, while the other two follow from (4.5) and the first equation of (4.6) and are hence approximations. The nature of the functions $a^n{}_{mik}$ and b_{ik} will be discussed in §8 below.

Because of (5.4), contraction of (5.2) by means of g^{ik} yields,

$$\frac{1}{\rho}\overline{\varpi_{,i} w^i} = \frac{1}{\rho}(\overline{\varpi w^i})_{,i} = 0. \tag{5.5}$$

This result is consistent with the correlation (3.3). For we may substitute the series in (5.1) into (3.3) and preserve the largest term; but the latter is smaller than the first term on the right-hand side of (5.2) by a factor of λ which is Taylor's scale of micro-turbulence [10, 8]; the second term on the right-hand side of (3.3) is also smaller than b_{ik} by an analogous factor. Hence the approximate form of (3.3), to the same degree of accuracy as in (5.2), is

$$\frac{1}{\rho}\overline{\varpi w_i} = 0. \tag{5.6}$$

This relation has also been proved to hold true for isotropic turbulence by von Kármán and Howarth [8].

By a similar process, we find from (3.2) that the triple correlation between the pressure gradient and two components of the velocity fluctuations is, to the same degree of approximation,

$$\frac{1}{\rho}(\overline{\varpi_{,i} w_k w_l} + \overline{\varpi_{,k} w_l w_i} + \overline{\varpi_{,l} w_i w_k}) = b^n{}_{mikl} U^m{}_{,n} + c_{ikl}, \tag{5.7}$$

where the forms of the tensors $b^n{}_{mikl}$ and c_{ikl} are given by respectively by

$$b^n{}_{mikl} = \frac{1}{2\pi}\iiint [(\overline{w'^n w_i w_k})_{,ml} + (\overline{w'^n w_k w_l})_{,mi} + (\overline{w'^n w_l w_i})_{,mk}] \frac{1}{r}\, dV',$$

$$c_{ikl} = \frac{1}{4\pi}\iiint [(\overline{u'^{mn} u_{ik}})_{,mnl} + (\overline{u'^{mn} u_{kl}})_{,mni} + (\overline{u'^{mn} u_{li}})_{,mnk}] \frac{1}{r}\, dV'. \tag{5.8}$$

Because of (4.6), the functions $b^n{}_{mikl}$ satisfy the rigorous conservation relation,

$$b^n{}_{nikl} = 0. \tag{5.9}$$

We shall discuss the general behaviour of the functions $b^n{}_{mikl}$ and c_{ikl} in §8.

6. Terms involving the decay of turbulence in the equations of double and triple correlation. To determine the terms in the decay of turbulence, it is necessary to know explicitly the double and triple velocity correlations between two adjacent points. Physically, the correlation functions between two near-by points must satisfy two conditions: first, they should become the velocity correlations at one point when the two points coincide; secondly, they should degenerate into the isotropic correlations when the flow obeys the condition of isotropy. By two adjacent points we mean that expansions of the double and triple correlation functions in terms of the coordinates ξ^i stop after the second and third powers of ξ^i, respectively. Furthermore, since only approximate expressions of the decay terms are required, conservation equations in the forms (4.4), (4.5) and (4.6) will suffice for the present purpose. In view of the property that the double correlation $\overline{w_i w'_n}$ satisfies the conservation relations (4.4) and (4.5) symmetrically with respect to the two indices i and n as a first approximation, it should also satisfy the supplementary condition that its expansion be symmetrical in the coordinates of P and P'.

The second order velocity correlation between two adjacent points that satisfies the above two conditions and the supplementary condition of symmetry can be expanded into powers of ξ^i in the form,

$$\overline{w_i w'_k} = \tfrac{1}{3}q^2\left\{\frac{A}{2\lambda^2}\,\xi_i\xi_k + \frac{\delta_{ik}}{2\lambda^2}\,B_{mn}\xi^m\xi^n + R_{ik}\left(1 + \frac{C_{mn}}{2\lambda^2}\,\xi^m\xi^n\right)\right.$$
$$\left. - \frac{G}{\lambda^2}\,(R_{il}\xi^l\xi_k + R_{kl}\xi^l\xi_i) + \frac{1}{4!\lambda^4}\,E_{ikjlmn}\xi^j\xi^l\xi^m\xi^n + \cdots\right\}, \tag{6.1}$$

where q is the mean magnitude of the velocity fluctuation, or the root-mean-square of the velocity fluctuation, defined by

$$q^2 = \overline{w_j w^j}, \tag{6.2}$$

and R_{ik} stands for

$$R_{ik} = \frac{3}{q^2}\,\overline{w_i w_k}. \tag{6.3}$$

The function λ is Taylor's scale of micro-turbulence, both q and λ being functions of the coordinates x^i of P; A, B_{mn}, C_{mn}, G and E_{ikjlmn} are all independent of ξ^i. The coefficients B_{mn} and C_{mn} are symmetric in m and n; E_{ikjlmn} is both symmetric in i and k and in the last four indices j, l, m and n, but is not symmetric in any one index of the first set of two and any one in the last set of four, e.g., it is not symmetric in i and j. Hence this tensor has $6\times 15 = 90$ independent components.

The form given in (6.1) for the correlation tensor $\overline{w_i w'_k}$ between two adjacent points is the most general linear combination of the products of the tensors $\xi_i\xi_k$, δ_{ik} and $w_i w_k$. The functions A, B_{ik}, C_{ik} and G will be assumed to be constants; it is not necessary to know the exact nature of the separate components of E_{ikjlmn} for our present purpose, but we shall assume for the time being that the invariant $E = g^{ik}g^{jl}g^{mn}E_{ikjlmn}$ is constant.

A question naturally arises as to whether the functions q^2 and λ^2 in (6.1), which vary with the coordinates, should be replaced by expressions which are symmetrical in the coordinates of P and P'. However, this is not essential, for both q^2 and λ^2 vary much more slowly than $\overline{w_i w'_k}$ as a function of ξ^i; since P and P' are close to each other, we may use their values at P as an approximation. Nevertheless, one must be careful with this approximation, whenever differentiation with respect to x'^i is involved.

The function $\overline{w_i w'_k}$ must satisfy the equation of continuity (4.4), or (4.5). By setting the coefficients of ξ^k and $\xi^j\xi^l\xi^m$ equal to zero separately, we find that

$$2A\delta_{ik} + B_{ik} + R^l{}_i C_{lk} - 5GR_{ik} - 3G\delta_{ik} = 0, \tag{6.4}$$

$$g^{ks}E_{ikslmn} = 0. \tag{6.5}$$

Since Eq. (6.4) is symmetric in the indices i and k, and since C_{lk} has been assumed to be a constant, we must have

$$C_{mn} = -C\delta_{mn}, \tag{6.6}$$

where C is a constant. On the other hand if C_{mn} depends upon the correlation tensor $\overline{w_i w_k}$, then it is possible to have the more general solution $C_{mn} = -C\delta_{mn} + DS_{mn}$, where S_{mn} is the inverse matrix of R_{mn} defined by $S^l{}_i R^k{}_l = \delta^k{}_i$. For the sake of simplicity, we choose D to be zero for the time being. Obviously, the number of independent equations in (6.5) is 30.

In order to give a simpler appearance to the final forms of the decay term in the equations of double correlation and of the equation of vorticity decay, we put

$$A = 1 + 4G, \qquad C = \tfrac{1}{3}(k - 4G). \tag{6.7}$$

The first equation amounts to a change of the factor λ, this factor being arbitrary; the change makes λ assume the same numerical value as Taylor's scale of microturbulence, when the correlation tensor obeys the condition of isotropy. The second equation in (6.7) only defines C in terms of a new constant k. Utilizing relations (6.4), (6.6) and (6.7), we put (6.1) into the form

$$\overline{w_i w'_k} = \overline{w_i w_k} + \frac{q^2}{3\lambda^2}\Big\{\tfrac{1}{2}(1 + 4G)\xi_i\xi_k - \tfrac{1}{2}\big[(2 + 5G)r^2 - \tfrac{1}{3}(k + 11G)R_{mn}\xi^m\xi^n\big]\delta_{ik}$$
$$- \tfrac{1}{6}(k - 4G)r^2R_{ik} - G(R_{il}\xi^l\xi_k + R_{kl}\xi^l\xi_i) + \frac{1}{4!\lambda^2}E_{ikjlmn}\xi^j\xi^l\xi^m\xi^n + \cdots\Big\}, \tag{6.8}$$

where the tensor E_{ikjlmn} satisfies the thirty linear equations (6.5). We shall see presently that, with the form of $\overline{w_i w'_k}$ given in (6.8), only the constant k will appear in the term involving the decay of turbulence (6.14), while only G will be present in the equation for the decay of vorticity (7.11).

For isotropic turbulence we have $\overline{w_i w_k} = \frac{1}{3}q^2\delta_{ik}$, and it is easy to verify that in (6.8) the terms in $\xi_i\xi_k$ and r^2 coincide with terms in the isotropic correlation tensor according to von Kármán and Howarth [8]. The validity of formula (6.8) and its properties can be subjected to experimental verification.

For the triple velocity correlation $\overline{w_i w_j w'_k}$ between two neighboring points, we have to assume a form which degenerates into $\overline{w_i w_j w_k}$ when the points coincide and becomes the triple correlation for isotropic turbulence when the condition of isotropy is satisfied by the flow. Since the expansion of the triple isotropic correlation function begins with the third powers of ξ^i, as shown by von Kármán and Howarth [8], the same must hold for the present general case. This expansion must satisfy the equation of continuity (4.6), and the final result obtained is

$$\overline{w_i w_j w'_k} = \overline{w_i w_j w_k} + \frac{Fq^3}{3!3\sqrt{3}\lambda^3}\left[2\xi_i\xi_j\xi_k - \frac{5}{2}(\delta_{ik}\xi_j + \delta_{jk}\xi_i)r^2 + \delta_{ij}\xi_k r^2\right] + \cdots. \tag{6.9}$$

This equation tells us that up to this degree of accuracy the correlation function $\overline{w_i w_j w'_k}$ is the sum of $\overline{w_i w_j w_k}$ and an isotropic correlation tensor; similarly, we have

$$\overline{w_i w_m' w'_n} = \overline{w_i w_m w_n} - \frac{Fq^3}{3!3\sqrt{3}\lambda^3}\left[2\xi_i\xi_m\xi_n - \frac{5}{2}(\delta_{mi}\xi_n + \delta_{ni}\xi_m)r^2 + \delta_{mn}\xi_i r^2\right] + \cdots. \tag{6.10}$$

In the above expression the relation (3.7) for isotropic turbulence has been utilized.

In the expansions of (6.9) and (6.10), we have introduced the further assumption that the triple correlation h can be expressed by [8]

$$h = \frac{F}{3!\lambda^3} r^3, \tag{6.11}$$

where λ is Taylor's scale of micro-turbulence and F is a numerical constant which may be different for different flows. This emphasizes the point that this length λ plays an important role, not only for double but also for triple correlations as well. The validity of this point should be tested experimentally.

Differentiating the correlation function (6.8) with respect to x'^s, we obtain

$$\frac{\partial}{\partial x'^s}\overline{w_i w'_k} = \overline{w_i \frac{\partial w'_k}{\partial x'^s}} = \frac{\partial}{\partial \xi^s}(\overline{w_i w'_k})_x$$

$$= \frac{q^2}{3\lambda^2}\Big\{\tfrac{1}{2}(1+4G)(\delta_{is}\xi_k + \delta_{ks}\xi_i) - \left[(2+5G)\xi_s - \tfrac{1}{3}(k+11G)R_{sn}\xi^n\right]\delta_{ik}$$

$$- \tfrac{1}{3}(k-4G)\xi_s R_{ik} - G(R_{is}\xi_k + R_{il}\delta_{ks}\xi^l + R_{ks}\xi_i + R_{kl}\delta_{is}\xi^l)$$

$$+ \frac{1}{3!\lambda^2}E_{ikjlms}\xi^j\xi^l\xi^m - \cdots\Big\}, \tag{6.12}$$

and furthermore, under the same approximation as in (4.6) where $\partial(\ )_\xi/\partial x^i$ is neglected, we get

$$\overline{\frac{\partial w_i}{\partial x^p}\frac{\partial w_k}{\partial x^s}} = \left[\frac{\partial}{\partial x^p}\left(\overline{w_i \frac{\partial w'_k}{\partial x'^s}}\right)\right]_{\xi=0} = -\left(\frac{\partial^2}{\partial\xi^p\partial\xi^s}\overline{w_i w'_k}\right)_{\xi=0}$$

$$= -\frac{q^2}{3\lambda^2}\Big\{\tfrac{1}{2}(1+4G)(\delta_{is}\delta_{kp} + \delta_{ks}\delta_{ip}) - \left[(2+5G)\delta_{sp} - \tfrac{1}{3}(k+11G)R_{sp}\right]\delta_{ik}$$

$$- \tfrac{1}{3}(k-4G)\delta_{sp}R_{ik} - G(R_{is}\delta_{kp} + R_{ip}\delta_{ks} + R_{ks}\delta_{ip} + R_{kp}\delta_{is})\Big\}. \tag{6.13}$$

Hence the term that represents the decay of turbulence in the equations of double correlation (1.6) is equal to

$$2\nu g^{mn}\overline{\frac{\partial w_i}{\partial x^m}\frac{\partial w_k}{\partial x^n}} = -\frac{2\nu}{3\lambda^2}(k-5)q^2 g_{ik} + \frac{2\nu k}{\lambda^2}\overline{w_i w_k}. \tag{6.14}$$

If we differentiate the triple correlation (6.9) with respect to the coordinates x'^m, the result is

$$\frac{\partial}{\partial x'^m}\overline{w_i w_k w'_l} = \overline{w_i w_k \frac{\partial w'_l}{\partial x'^m}} = \frac{\partial}{\partial \xi^m}(\overline{w_i w_k w'_l})_x.$$

Similarly, to the same order of approximation as in (6.13) the following relation is true:

$$\frac{\partial}{\partial x^n}\overline{w_i w_k \frac{\partial w'_l}{\partial x'^m}} = \overline{w_i \frac{\partial w_k}{\partial x^n}\frac{\partial w'_l}{\partial x'^m}} + \overline{w_k \frac{\partial w_i}{\partial x^n}\frac{\partial w'_l}{\partial x'^m}} = -\frac{\partial^2}{\partial \xi^m \partial \xi^n}(\overline{w_i w_k w'_l})_x.$$

This equation and formula (6.9) then yield

$$\overline{w_i \frac{\partial w_k}{\partial x^n}\frac{\partial w_l}{\partial x^m}} + \overline{w_k \frac{\partial w_i}{\partial x^n \partial x^m}\frac{\partial w_l}{}} = -\left(\frac{\partial^2}{\partial \xi^m \partial \xi^n}\overline{w_i w_k w'_l}\right)_{\xi=0} = 0. \quad (6.15)$$

Cyclic permutation of the indices i, k, l in (6.15) gives rise to two similar relations; the sum of the three is identically zero, which shows that the term analogous to the decay of turbulence in the equations of triple correlation vanishes in general:

$$2\nu g^{mn}[\overline{w_{i,m}w_{k,n}w_l} + \overline{w_{k,m}w_{l,n}w_i} + \overline{w_{l,m}w_{i,n}w_k}] = 0. \quad (6.16)$$

7. The equation of vorticity decay. Since Taylor's scale of micro-turbulence λ plays a very important role in the decay of turbulence, it is necessary to find the equation which governs the behaviour of this fundamental length. This equation is provided by the decay of vorticity. The root-mean-square of the vorticity fluctuation $(\overline{\omega^2})^{1/2}$ satisfies the equation

$$\overline{\omega^2} = \tfrac{1}{2}g^{mp}g^{ns}\overline{\omega_{mn}\omega_{ps}}, \quad (7.1)$$

where ω_{mn} is the antisymmetrical tensor defined by (1.5). It is not difficult to derive the equation satisfied by $\overline{\omega^2}$ from (1.4) directly. However, this procedure would be too lengthy and we shall pursue an alternative course.

We notice that

$$g^{mp}g^{ns}\overline{\omega_{mn}\omega_{ps}} = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2}(w_{m,n} - w_{n,m})(w^m{}_{,s}g^{ns} - w^n{}_{,p}g^{mp})dt$$

$$= 2(\overline{w_{m,n}w^m{}_{,s}}g^{ns} - \overline{w^n{}_{,m}w^m{}_{,n}}). \quad (7.2)$$

On the other hand, to the same order of approximation as in (6.12) and (6.13), the following expressions are true:

$$g^{ns}\overline{w_{m,n}w^m{}_{,s}} = -(\nabla^2_\xi \overline{w_m w'^m})_{\xi=0}, \qquad \overline{w^n{}_{,m}w^m{}_{,n}} = -\left(\frac{\partial^2}{\partial \xi^m \partial \xi^n}\overline{w^m w'^n}\right)_{\xi=0} = 0,$$

where ∇^2_ξ stands for the Laplacian operator in the variables ξ^i. It then follows that

$$\overline{\omega^2} = -(\nabla^2_\xi \overline{w_m w'^m})_{\xi=0}. \quad (7.3)$$

Our next step is to derive the differential equation satisfied by $(\nabla^2_\xi \overline{w_m w'^m})_{\xi=0}$.

From the equation of turbulent fluctuation at the point P', which can be written in the form

$$\frac{\partial w'_k}{\partial t} + U'^j w'_{k,j} + w'^j w'_{k,j} + w'^j U'_{k,j} = -\frac{1}{\rho}\varpi'_{,k} - \frac{1}{\rho}\tau'^j{}_{k,j} + \nu\nabla'^2 w'_k, \quad (7.4)$$

we derive the equation satisfied by the general double correlation function:

$$\frac{\partial}{\partial t}\overline{w_i w'_k} + U^j(\overline{w_i w'_k})_{,j} + U'^j(\overline{w_i w'_k})'_{,j} + (\overline{w^j w_i w'_k})_{,j} + (\overline{w'^j w_i w'_k})'_{,j}$$
$$+ \overline{w'_k w^j} U_{i,j} + \overline{w_i w'^j} U'_{k,j}$$
$$= -\frac{1}{\rho}(\overline{\varpi w'_k})_{,i} - \frac{1}{\rho}(\overline{\varpi' w_i})'_{,k} + \nu\nabla^2(\overline{w_i w'_k}) + \nu\nabla'^2(\overline{w_i w'_k}), \tag{7.5}$$

where the covariant derivatives $(\ )_{,j}$ and $(\ )'_{,j}$ are taken with respect to the variables x^j and x'^j, respectively.

In Eq. (7.5) we next replace x^i and x'^i by the two new sets of variables x^i and ξ^i by use of (4.2), and neglect terms involving the partial derivatives with respect to x^i when ξ^i are held constant, except for the term $U^j\partial(\ )_\xi/\partial x^j$; this exception is made because U^j is large when compared with w^k. Since we are only interested in the correlation functions for two adjacent points, we can write

$$(\overline{w'^j w_i w'_k})'_{,j} = -\frac{\partial}{\partial \xi^j}(\overline{w^j w'_i w_k})_x,$$

as in the case of isotropic turbulence (3.7). With all these approximations in view, Eq. (7.5) in rectangular coordinates then becomes

$$\frac{\partial}{\partial t}\overline{w_i w'_k} + U^j\frac{\partial}{\partial x^j}(\overline{w_i w'_k})_\xi - U^j\frac{\partial}{\partial \xi^j}(\overline{w_i w'_k})_x + U'^j\frac{\partial}{\partial \xi^j}(\overline{w_i w'_k})_x$$
$$-\frac{\partial}{\partial \xi^j}(\overline{w^j w_i w'_k} + \overline{w^j w'_i w_k})_x + \overline{w'_k w^j}\frac{\partial U_i}{\partial x^j} + \overline{w_i w'^j}\frac{\partial U'_k}{\partial \xi^j}$$
$$= \frac{1}{\rho}\frac{\partial}{\partial \xi^i}(\overline{\varpi w'_k})_x - \frac{1}{\rho}\frac{\partial}{\partial \xi^k}(\overline{\varpi' w_i})_x + 2\nu\nabla^2_\xi(\overline{w_i w'_k})_x. \tag{7.6}$$

For two adjacent points the power series expansion of $\overline{\varpi w'_k}$ in ξ^i is in odd powers of ξ^i. Hence, by interchanging the two points P and P', we should have

$$\overline{\varpi w'_i} = -\overline{\varpi' w_i}. \tag{7.7}$$

Consequently (7.6) is essentially symmetric in the indices i and k.

Next, let us contract the indices i and k in (7.6). As in (4.4), $\overline{\varpi w'_k}$ should satisfy rigorously the equation of continuity,

$$\frac{\partial}{\partial \xi^k}(\overline{\varpi w'^k})_x = 0. \tag{7.8}$$

The result of this contraction then becomes

$$\frac{\partial}{\partial t}\overline{w_k w'^k} + U^j\frac{\partial}{\partial x^j}(\overline{w_k w'^k})_\xi - U^j\frac{\partial}{\partial \xi^j}(\overline{w_k w'^k})_x + U'^j\frac{\partial}{\partial \xi^j}(\overline{w_k w'^k})_x$$
$$-2\frac{\partial}{\partial \xi^j}(\overline{w^j w_k w'^k})_x + \overline{w'_k w^j}\frac{\partial U^k}{\partial x^j} + \overline{w_k w'^j}\frac{\partial U'^k}{\partial \xi^j} = 2\nu\nabla^2_\xi(\overline{w_k w'^k})_x. \tag{7.9}$$

Let us operate upon (7.9) with the Laplacian operator ∇^2_ξ and then set $\xi^i=0$, denoting $(\ )_{\xi=0}$ by $(\ )_0$ for smiplicity. The resulting equation is,

$$\frac{\partial}{\partial t}(\nabla^2_\xi \overline{w_k w'^k})_0 + U^j \frac{\partial}{\partial x^j}(\nabla^2_\xi \overline{w_k w'^k})_0 + 2U^j_{,m} g^{mn}\left(\frac{\partial^2}{\partial\xi^j\partial\xi^n}\overline{w_k w'^k}\right)_0$$
$$-2\left(\nabla^2_\xi \frac{\partial}{\partial \xi^j}\overline{w^j w_k w'^k}\right)_0 + 2U_{k,j}(\nabla^2_\xi \overline{w^j w'^k})_0 = 2\nu(\nabla^4_\xi \overline{w_k w'^k})_0. \quad (7.10)$$

It is to be noted that in the above equation we have neglected the term $\overline{w_k w^j}\nabla^2_x U^k_{,j}$, which is smaller than the term $U_{k,j}(\nabla^2_\xi \overline{w^j w'^k})_0$ by a factor which is the square of the ratio of λ to a macroscopic length. Equation (7.10) also follows from the equation of vorticity fluctuation (1.4) directly as mentioned before.

By substituting into (7.10) the explicit forms of the correlation functions $\overline{w_i w'_k}$ and $\overline{w_i w_j w'_k}$ for two adjacent points given in (6.8) and (6.9), respectively, and then setting $\xi^l=0$, we obtain the equation of vorticity decay,

$$5\frac{\partial}{\partial t}\left(\frac{q^2}{\lambda^2}\right) + 5U^j\frac{\partial}{\partial x^j}\left(\frac{q^2}{\lambda^2}\right) - \frac{14G}{\lambda^2}U_{i,k}\overline{w^i w^k} - \frac{70F}{3\sqrt{3}}\frac{q^3}{\lambda^3} = -\frac{2\nu}{3}E\frac{q^2}{\lambda^4}, \quad (7.11)$$

in which E is defined as before,

$$E = g^{ik}g^{jl}g^{mn}E_{ikjlmn}. \quad (7.12)$$

We assume that both E and F are constants which may be different for flows with different Reynolds numbers. In deriving equation (7.11), the equation of continuity $U^j_{,j}=0$ for the mean motion has been utilized. It is also readily verifiable that (7.11) agrees with von Kármán's equation of vorticity decay for isotropic turbulence [8].

8. Nature of the correlation integrals and the final forms of the dynamical equations of correlation. Up to the present the only remaining uncertain quantities in the equations of the double and triple correlations (1.6) and (1.7) are the correlation integrals, a^n_{mik}, and b_{ik} in (5.3), b^n_{mikl} and c_{ikl} of (5.8), and the quadruple velocity correlation $\overline{w_j w_i w_k w_l}$. Let us examine the correlation integrals first. The function a^n_{mik} defined in (5.3), for example, would be uniquely determined if the double correlation $\overline{w_i w'_k}$ were known. But unfortunately the equation of continuity (4.4) and the general dynamical equation of double correlation (7.6) are insufficient to yield a definite solution for $\overline{w_i w'_k}$, because of the presence of the triple correlation $\overline{w_i w_j w'_k}$ in (7.6).

On the other hand, although the integrand functions of the four kinds of correlation integrals are not known, we are dealing primarily with the integrals themselves and they can only vary slowly with the coordinates involved. This argument can be understood, if we recall that the correlation functions $\overline{w_i w'_n}$, $\overline{w_i w_k w'_n}$, $\overline{w_i w'_m w'^n}$ and $\overline{u_{ik} u'_{mn}}$ under the integral signs only change slowly when both the point P and the point of integration P' undergo a rigid body translation, and that they vary rapidly when the relative displacement of the two points changes. This rapidly varying part of the functions is integrated away, leaving the slowly varying part behind. The neglecting of the term $\partial(\overline{w'_n w^i})_\xi/\partial x^i$ against $\partial(\overline{w'_n w^i})_x/\partial\xi^i$ in (4.5) also follows from this interpretation.

There is another mathematical reason for the fact that the four kinds of integrals are slowly varying functions of the coordinates. If, for instance, we differentiate with respect to x^s the quantities c_{ikl} defined in (5.8), we find that

$$\frac{\partial c_{ikl}}{\partial x^s} = \frac{1}{4\pi}\int\int\int \left\{ \frac{\partial}{\partial x^s}\left(\left[(\overline{u'^{mn}u_{ik}})_{,mnl} + (\overline{u'^{mn}u_{kl}})_{,mni} + (\overline{u'^{mn}u_{li}})_{,mnk} \right] \frac{1}{r} \right)_\xi \right.$$
$$\left. - \frac{\partial}{\partial \xi^s}\left(\left[(\overline{u'^{mn}u_{ik}})_{,mnl} + (\overline{u'^{mn}u_{kl}})_{,mni} + (\overline{u'^{mn}u_{li}})_{,mnk} \right] \frac{1}{r} \right)_x \right\} dV'. \qquad (8.1)$$

The first part of the integrand function is small when compared with the second, and the second can be transformed into a surface integral on the boundary of the fluid by means of the usual divergence theorem of vector analysis. If the point P is not very close to the surface, this surface integral is negligible on the ground that the correlation function $\overline{u'^{mn}u_{ik}}$ and its derivatives between P, the point in the interior of the fluid, and P', the point of integration on the boundary, are negligible.

Since the correlation integrals are slowly varying functions of the coordinates, we shall expand them as powers of the coordinates used in the special problems to be solved. From kinematic considerations, the integrands of the integrals may furthermore contain powers of q, the root-mean-square of the velocity fluctuation, as factors. Both theory and experiment at present do not assure us of the exact dependence of this factor. Nevertheless, so far as the mean velocity distribution is concerned, this uncertainty is probably not important, as we shall see in the problem of pressure flow between two parallel infinite planes [9].

By substituting into Eqs. (1.6) and (1.7) the approximate forms of the four correlation integrals from (5.2) and (5.7), and the decay terms (6.14) and (6.16), we obtain finally

$$-\frac{1}{\rho}\frac{\partial \tau_{ik}}{\partial t} - \frac{1}{\rho}(U_{i,j}\tau^j{}_k + U_{k,j}\tau^j{}_i) - \frac{1}{\rho}U^j\tau_{ik,j} + (\overline{w^jw_iw_k})_{,j}$$
$$= -a^n{}_{mik}U^m{}_{,n} - b_{ik} - \frac{\nu}{\rho}\nabla^2\tau_{ik} + \frac{2\nu}{3\lambda^2}(k-5)q^2g_{ik} - \frac{2\nu k}{\lambda^2}\overline{w_iw_k}, \qquad (8.2)$$

$$\frac{\partial}{\partial t}\overline{w_iw_kw_l} + U_{i,j}\overline{w^jw_kw_l} + U_{k,j}\overline{w^jw_lw_i} + U_{l,j}\overline{w^jw_iw_k} + U^j(\overline{w_iw_kw_l})_{,j} + (\overline{w^jw_iw_kw_l})_{,j}$$
$$= -b^n{}_{mikl}U^m{}_{,n} - c_{ikl} + \frac{1}{\rho^2}(\tau^j{}_{i,j}\tau_{kl} + \tau^j{}_{k,j}\tau_{li} + \tau^j{}_{l,j}\tau_{ki}) + \nu g^{mn}(\overline{w_iw_kw_l})_{,mn}. \qquad (8.3)$$

In the second set of equations we notice that the term involving the quadruple correlation is actually smaller than the terms $b^n{}_{mikl}U^m{}_{,n}$ and c_{ikl} which form the correlation between the pressure gradient and two components of velocity fluctuation. This is due to the fact that the term $(\overline{w^jw_iw_kw_l})_{,j}$ is equal to a velocity fluctuation raised to the fourth power and divided by a macroscopic length, while on the other hand c_{ikl} is, from its definition (5.8), of the order of a velocity fluctuation raised to the fourth power and divided by a length which has the same order of magnitude as Taylor's scale of micro-turbulence. The permissibility of neglecting the terms $(\overline{w^jw_iw_kw_l})_{,j}$ and $\rho^{-2}\tau^j{}_{i,j}\tau_{kl}$ as a first approximation, for instance in the problem of pressure flow between two parallel infinite planes [9], can be regarded as a justification of the above approximation and its associated interpretation.

We must not forget that the other dynamical equations necessary for the solution of a turbulence problem are the equations of mean motion (1.1) and the equation of vorticity decay (7.11).

9. Conclusion and summary. It is now not difficult to see that the foregoing development is essentially a method of successive approximation to the solution of the turbulence problem. In the initial approximation we have the well-known Reynolds' equations of mean motion which contain the unknown apparent stress. From the mathematical point of view the momentum and vorticity transport theories connect this stress with the mean velocity by physical arguments, in order to make the mean velocity distribution determinate.

The next approximation in solving the given turbulence problem is to use the equations of mean motion and of double correlation by making certain approximations to the triple velocity correlation in the equations. This procedure has been followed in the determination of the velocity distributions in jets [3] and wakes [4], where free turbulence predominates; for the triple correlations we use their values at the centers of the flows as an approximation. The mean velocity distributions thus obtained agree with the experimental observations very well over large portions of the flows.

In the third approximation to the solution of the problem we have to solve the equations of mean motion and of both the double and triple correlations simultaneously by assuming approximations for the quadruple correlations. It is obvious that this process of forming the differential equations of the correlations out of the equations of turbulent fluctuation can be generalized to higher orders. Fortunately, as in the problem of pressure flow through a channel [9] where a wall is present, we can stop at the equations of triple correlation and neglect the quadruple correlations as an approximation, so that the solution of the problem is not too unnecessarily complicated from the theoretical point of view. As we shall see, the solution of this particular problem holds true in all parts of the channel, if all the equations of mean motion and of double and triple correlation are used. On the other hand, the solution for the mean velocity based upon the equations of mean motion and of double correlation by using the value of the triple correlation in the center of the channel as in jets and wakes, is only valid in the central part of the channel, and fails when the wall of the channel is approached. This brings up incidentally the important role played by the triple correlation in such problems.

In order to see more clearly how the equations of double and triple correlation in the forms (8.2) and (8.3) and the equation of vorticity decay (7.11) are derived from the equations of turbulent fluctuation, it might be of interest to sum up the conditions and approximations under which they are valid. They are listed below:

(1) The velocity correlation between a point in the interior of the fluid and another on the boundary is negligible. This excludes the immediate neighborhood of the boundary of the fluid as a region of application of the theory.

(2) The variation of the mean velocity is small as compared with the correlation function between two distinct points when the relative displacement between the points changes, so that the higher order terms in the series (5.1) and similar series may be dropped.

(3) The second and third order velocity correlations between two adjacent points

are expansible as power series in ξ^i/λ with the terms that do not contain ξ^i proportional to the Reynolds stress at the points. This brings out the point that Taylor's scale of micro-turbulence λ plays an equally important role for both the double and triple velocity correlations.

(4) The slowly varying nature of the functions $a^n{}_{mik}$, b_{ik}, $b^n{}_{mikl}$ and c_{ikl} with the coordinates, and its physical interpretation, have been explained in the preceding section.

With the advance of modern experimental technique the above four conditions and their theoretical consequences, as presented here, can all be tested by direct experimental observation. The less certain part of the theory lies probably in the discussions in §8 of the slowly varying nature of the correlation integrals with the coordinates; this perhaps could be improved if more accurate experimental evidence were available.

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