ON A CLASS OF DIFFERENTIAL EQUATIONS IN MECHANICS OF CONTINUA*

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1. Introduction. This paper is concerned with partial differential equations of the form

$$\begin{aligned} u_x &= \tau_1(y)v_y, \\ u_y &= -\tau_2(y)v_x, \end{aligned} \tag{1.1}$$

and with the second order equations

$$\begin{aligned} \frac{u_{xx}}{\tau_1} + \left(\frac{u_y}{\tau_2}\right)_y &= 0, \\ \tau_2 v_{xx} + (\tau_1 v_y)_y &= 0, \end{aligned} \tag{1.2}$$

which are obtained from (1.1) by eliminating either v or u. Here $u=u(x, y)$, $v=v(x, y)$, and subscripts x, y denote partial derivatives. Equations of this form are frequently found in problems of mechanics of continua (cf. the examples in sections 5–7).

The system (1.1) possesses the same structure as the Cauchy-Riemann equations

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \tag{1.3}$$

connecting the real and the imaginary parts of an analytic function of $x+iy$. This similarity suggests an integration theory similar in pattern to that of the complex function theory. The fundamentals of such theory are presented in this paper. (A more elaborate mathematical treatment, containing all proofs, will be published elsewhere). The theory will be illustrated by some physical examples. In treating these examples our aim is not to obtain new results in mechanics but rather to present known facts from a simpler and more unified point of view.

In what follows we suppose that the coefficients τ_i $(i=1, 2)$ are *positive analytic* functions of the real variable y. Then the equations (1.2) are of

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elliptic type with analytic coefficients. Therefore u and v will be analytic functions of the real variables x and y.

2. Σ-monogenic functions. If u and v are two conjugate harmonic functions, (i.e. solutions of (1.3)), then $f=u+iv$ is an analytic function of the complex variable $z=x+iy$. From u and v we can obtain two other pairs of conjugate harmonic functions, U, V and u', v', by setting

$$U + iV = \int_{z_0}^{z} f dz, \qquad u' + iv' = f'(z) = \frac{df}{dz},$$

i.e.

$$U = \int_{z_0}^{z} u dx - v dy, \qquad V = \int_{z_0}^{z} v dx + u dy$$

and

$$u' = u_x = v_y, \qquad v' = v_x = -u_y.$$

This procedure can be extended to equations (1.1). Let u and v be a pair of solutions of (1.1). We define the functions

$$U = \int_{z_0}^{z} u dx - \tau_2 v dy, \qquad V = \int_{z_0}^{z} v dx + \frac{u}{\tau_1} dy \tag{2.1}$$

$$u' = u_x = \tau_1 v_y, \qquad v' = v_x = -\frac{u_y}{\tau_2}. \tag{2.2}$$

In (2.1) the integration is extended from an arbitrary fixed point $x_0+iy_0=z_0$ to a variable point $x+iy=z$. By virtue of (1.1) these line integrals do not depend upon the path, but only upon the points z, z_0.

We have

$$U_x = u, \quad U_y = -\tau_2 v, \qquad V_x = v, \quad V_y = \frac{u}{\tau_1},$$

and

$$u'_x = \tau_1 v_{xy}, \quad u'_y = u_{xy}, \qquad v'_x = -\frac{u_{xy}}{\tau_2}, \quad v'_y = v_{xy}.$$

Thus U, V and u, v are solutions of (1.1).

It is convenient to consider

$$f = u + iv$$

as a function of the complex variable $z=x+iy$. If u and v satisfy (1.1) we shall call f a *Σ-monogenic function,* Σ denoting the matrix of the coefficients of (1.1); i.e.,

$$\Sigma = \begin{Vmatrix} 1 & \tau_1(y) \\ 1 & \tau_2(y) \end{Vmatrix}.$$

We have shown that

$$F = U + iV \quad \text{and} \quad f^{\backprime} = u^{\backprime} + iv^{\backprime} \tag{2.3}$$

are also Σ-monogenic functions. We call them the *Σ-integral* and the *Σ-derivative* of f respectively, and write

$$F = \int_{z_0}^{z} f d_\Sigma z, \qquad f^{\backprime} = \frac{d_\Sigma f}{d_\Sigma z}. \tag{2.4}$$

The definition of higher Σ-derivatives is obvious. We write

$$f^{[0]} = f, \quad f^{[1]} = f^{\backprime}, \quad f^{[2]} = f^{\backprime\backprime}, \quad \text{etc.}$$

A comparison of the formulae (2.1)–(2.4) shows that Σ-integration and Σ-differentiation are inverse processes.

By applying Σ-integration to constants we can obtain an unlimited number of *particular solutions* of (1.1). The function $f=1$ is Σ-monogenic, for $u=1$, $v=0$ is a solution of (1.1). Therefore

$$Z^{(1)}(z) = \int_0^z 1 d_\Sigma z = x + i\int_0^y \frac{dy}{\tau_1}$$

is Σ-monogenic, i.e., $u=x$, $v=\int_0^y dy/\tau_1$ is a solution of (1.1). Next,

$$Z^{(2)}(z) = 2\int_0^z Z^{(1)} d_\Sigma z = x^2 - 2\int_0^y \tau_2 dy \int \frac{dy}{\tau_1} + 2ix\int_0^y \frac{dy}{\tau_1}$$

is Σ-monogenic and so are

$$Z^{(3)}(z) = 3\int_0^z Z^{(2)} d_\Sigma z, \qquad Z^{(4)}(z) = 4\int_0^z Z^{(3)} d_\Sigma z, \cdots.$$

It is not difficult to find a general formula for $Z^{(n)}(z)$.[1] We set $Z^{(0)}=1$ and define

$$\begin{gathered} Y^{(0)}(y) = 1, \qquad Y^{(1)}(y) = \int_0^y \frac{dy}{\tau_1}, \qquad Y^{(2)}(y) = 2\int_0^y \tau_2 Y^{(1)}(y) dy, \\ Y^{(3)}(y) = 3\int_0^y \frac{1}{\tau_1} Y^{(2)}(y) dy, \cdots. \end{gathered} \tag{2.5}$$

Then

$$Z^{(n)}(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu i^{n-\nu} Y^{(n-\nu)}(y). \tag{2.6}$$

[1] Note that the superscript (n) does not denote differentiation. For the gas-dynamical equations written in the form (1.1), where $\tau_1 = 1$, particular solutions corresponding to the formal powers $Z^{(n)}$ have been obtained independently of us by Professor S. Bergman (see footnote 14), and for the imaginary parts by Dr. A. Vaszonyi.

By repeated Σ-integration of the function $f=i$, we obtain another set of Σ-monogenic functions, which we denote by

$$i \cdot Z^{(n)}(z).$$

We have

$$i \cdot Z^{(n)}(z) = i \sum_{\nu=0}^{n} \binom{n}{\nu} x^{\nu} i^{n-\nu} Y^{*(n-\nu)}(y), \tag{2.7}$$

where

$$Y^{*(0)}(y) = 1, \quad Y^{*(1)}(y) = \int_0^y \tau_2 dy, \quad Y^{*(2)} = 2\int_0^y \frac{1}{\tau_1} Y^{*(1)} dy, \cdots . \tag{2.8}$$

Finally, for any complex constant $a=\alpha+i\beta$ we set

$$a \cdot Z^{(n)}(z) = \alpha Z^{(n)}(z) + \beta \{ i \cdot Z^{(n)}(z) \}. \tag{2.9}$$

Clearly

$$\frac{d_\Sigma}{d_\Sigma z} \{ a \cdot Z^{(n)} \} = n \, a \cdot Z^{(n-1)}. \tag{2.10}$$

From the "formal powers" (2.9) we can construct new particular solutions. Obviously a "formal polynomial" of the nth degree

$$f(z) = a_0 + a_1 \cdot Z^{(1)}(z) + \cdots + a_n \cdot Z^{(n)}(z), \qquad a_n \neq 0,$$

is a Σ-monogenic function; i.e., its real and imaginary parts satisfy (1.1). It can be shown that there always exists a formal polynomial of the nth degree which takes prescribed values $A_1, A_2, \cdots, A_{n+1}$ at $n+1$ prescribed points $z_1, \cdots, z_{n+1}$.

A "formal power series"

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot Z^{(n)}(z) \tag{2.11}$$

represents a Σ-monogenic function provided the series converges uniformly and absolutely for sufficiently small values of $|z|$. It can be shown that *any function which is Σ-monogenic around the origin can be represented in the form* (2.11). The coefficients are given by a "Taylor formula":

$$n! a_n = f^{[n]}(0). \tag{2.12}$$

If in defining the formal powers we would extend the integration not from 0 to z but from another fixed point z_0, we would obtain another set of particular solutions, which we denote by

$$a \cdot \underset{z_0}{Z^{(n)}}(z).$$

Any Σ-monogenic function defined in the neighborhood of z_0, can be represented in the form

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot \underset{z_0}{Z^{(n)}}(z). \tag{2.13}$$

The same method can be applied to the more general equations

$$\begin{aligned} \sigma_1(x)u_x &= \tau_1(y)v_y, \\ \sigma_2(x)u_y &= -\tau_2(y)v_x \end{aligned} \tag{2.14}$$

($\sigma_i > 0$, $\tau_i > 0$, $i = 1, 2$). We denote the matrix of the coefficients of (2.14) by Σ,

$$\Sigma = \left\| \begin{matrix} \sigma_1 & \tau_1 \\ \sigma_2 & \tau_2 \end{matrix} \right\|,$$

and call a function $f = u + iv$ Σ-monogenic if u and v satisfy (2.14). The Σ-integral, $F = U + iV$, and the Σ-derivative, $f^{\backprime} = u^{\backprime} + iv^{\backprime}$, of f are defined by

$$U = \int_{z_0}^{z} \sigma_2 u dx - \tau_2 v dy, \qquad V = \int_{z_0}^{z} \frac{v}{\sigma_1} dx + \frac{u}{\tau_1} dy,$$

and

$$u^{\backprime} = \sigma_1 u_x = \tau_1 v_y, \qquad v^{\backprime} = \frac{v_x}{\sigma_2} = -\frac{u_y}{\tau_2}.$$

U, V and $u^{\backprime}$, $v^{\backprime}$ satisfy not (2.14) but the *associated equations*

$$\begin{aligned} \frac{u_x}{\sigma_2(x)} &= \tau_1(y)v_y \\ \frac{u_y}{\sigma_1(x)} &= -\tau_2(y)v_x. \end{aligned} \tag{2.15}$$

Thus F and $f^{\backprime}$ are Σ'-monogenic, where

$$\Sigma' = \left\| \begin{matrix} \dfrac{1}{\sigma_2} & \tau_1 \\ \dfrac{1}{\sigma_1} & \tau_2 \end{matrix} \right\|.$$

However, $\Sigma'' = \Sigma$, so that the Σ'-integral of F and the Σ'-derivative of $f^{\backprime}$ are Σ-monogenic.

It is clear in which way we must change the definition of the formal powers. We set

$$Z^{(1)} = \int 1 d_{\Sigma'} z, \qquad Z^{(2)} = 2\int \tilde{Z}^{(1)} d_{\Sigma'} z \quad \text{where} \quad \tilde{Z}^{(1)} = \int 1 d_{\Sigma} z,$$

$$Z^{(3)} = 3\int \tilde{Z}^{(2)} d_{\Sigma'} z \quad \text{where} \quad \tilde{Z}^{(2)} = 2\int Z^{(1)} d_{\Sigma} z, \cdots,$$

and similarly for $i \cdot Z^{(n)}$.

Remark. If in (1.1) τ_1 and τ_2 are of *opposite sign*, the equations (1.2) are of *hyperbolic type*. The definition of Σ-monogenic functions, Σ-integrals and derivatives, formal powers and formal power series remains the same. However, it is *not* true that u and v are necessarily analytic functions of x and y. Neither is it true that *all* Σ-monogenic functions can be represented in the form (2.13).

The integrals defining $Z^{(n)}$ and $i \cdot Z^{(n)}$ must not necessarily converge when τ_1 or τ_2 vanishes at z_0 or at z. If they do converge, they represent Σ-monogenic functions. In this way it is possible to obtain particular solutions of partial differential equations which are of elliptic type in one part of the plane and of hyperbolic in another.

3. Correspondence between Σ-monogenic and analytic functions. Let

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n \tag{3.1}$$

be an analytic function of z. We define the Σ-monogenic function

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot Z^{(n)}(z) \tag{3.2}$$

and say that *f corresponds to φ at the origin.*[2]

It can be shown that series (3.2) converges in some neighborhood of the origin. If (3.1) converges everywhere, so does (3.2). If (at the origin) the Σ-monogenic functions f and g correspond to φ and ψ, then $f+g$ corresponds to $\varphi+\psi$, $f^{[n]}$ to $\varphi^{(n)}$, and αf to $\alpha\varphi$, α being a real constant.

The concept of corresponding functions may be of use in discussing physical problems. Assume that some physical phenomenon is described by equations of the form (1.1). Often a simplifying assumption (for instance: the assumption of incompressibility) leads to Cauchy-Riemann equations. Suppose we possess an interesting or typical solution of the simplified problem. It will be given by an analytic function $\zeta = \varphi(z)$. We may expect that the solution of the original more complicated problem given by

$$\zeta = f(z), \ f \text{ corresponding to } \varphi,$$

will be of the same general character.

[2] Thus the formal powers "correspond" to ordinary powers.

We define the Σ-monogenic exponential and trigonometric functions (depending upon a real parameter α)

$$E(\alpha, z), \quad S(\alpha, z), \quad C(\alpha, z), \quad i\cdot E(\alpha, z), \quad i\cdot S(\alpha, z), \quad i\cdot C(\alpha, z) \tag{3.3}$$

as the functions which (at the origin) correspond to the analytic functions

$$e^{\alpha z}, \quad \sin \alpha z, \quad \cos \alpha z, \quad ie^{\alpha z}, \quad i \sin \alpha z, \quad i \cos \alpha z.$$

We have, for instance,

$$E(\alpha, z) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} Z^{(n)}(z), \qquad i\cdot S(\alpha, z) = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!}\, i\cdot Z^{(2n+1)}(z).$$

The function $C(\alpha, z)+i\cdot S(\alpha, z)$ corresponds to $e^{i\alpha z}$; $E(\alpha, z)+E(-\alpha, z)$, $E(\alpha, z)-E(-\alpha, z)$ correspond to 2 cosh αz and 2 sinh αz.

It is worthwhile to divide the functions (3.3) into their real and imaginary parts. A simple calculation shows that

$$\begin{aligned} E(\alpha, z) &= e^{\alpha x}[c(\alpha, y) + i\, s(\alpha, y)] \\ S(\alpha, z) &= \sin \alpha x\, ch(\alpha, y) + i \cos \alpha x\, sh(\alpha, y) \\ C(\alpha, z) &= \cos \alpha x\, ch(\alpha, y) - i \sin \alpha x\, sh(\alpha, y) \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} s(\alpha, y) &= \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!} Y^{(2n+1)}(y), & sh(\alpha, y) &= \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!} Y^{(2n+1)}(y) \\ c(\alpha, y) &= \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n+1)!} Y^{(2n)}(y), & ch(\alpha, y) &= \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} Y^{(2n)}(y). \end{aligned} \tag{3.5}$$

Note that the series (3.5) converge for all real values of y and for all complex values of α, so that we may write

$$sh(\alpha, y) = -i\, s(i\alpha, y), \qquad ch(\alpha, y) = c(i\alpha, y). \tag{3.6}$$

Similarly we have

$$\begin{aligned} i\cdot E(\alpha, z) &= e^{\alpha x}[-s^*(\alpha, y) + ic^*(\alpha, y)] \\ i\cdot S(\alpha, z) &= -\cos \alpha x\, sh^*(\alpha, y) + i \sin \alpha x\, ch^*(\alpha, y) \\ i\cdot C(\alpha, z) &= \sin \alpha x\, sh^*(\alpha, y) + i \cos \alpha x\, ch^*(\alpha, y), \end{aligned} \tag{3.7}$$

where the functions s^*, c^*, sh^* and ch^* are defined in the same way as in (3.5), $Y^{(n)}$ being replaced by $Y^{*(n)}$.

The right hand sides of (3.4) and (3.7) have the form $\Phi_1(x)\Psi_1(y) + i\Phi_2(x)\Psi_2(y)$. Thus we see that the real and the imaginary parts of the Σ-monogenic exponential and trigonometric functions coincide with the functions which we would obtain by solving equations (1.2) by the method of the *separation of variables*. Therefore the functions (3.3) can be used in solving certain boundary value problems.

The functions $E, \cdots, i\cdot C$ possess many properties analogous to those of ordinary exponential and trigonometric functions. For instance, the Σ-differential equations

$$E^{\backprime} = \alpha E, \qquad S^{\backprime} = \alpha C, \qquad C^{\backprime} = -\alpha S. \tag{3.8}$$

hold.

In the same way we could define correspondence at an arbitrary point z_0 and discuss the functions $\underset{z_0}{E}(\alpha, z), \underset{z_0}{S}(\alpha, z), \cdots$, which at z_0 correspond to the analytic functions $e^{\alpha(z-z_0)}$, $\sin \alpha(z-z_0), \cdots$.

In some special cases the formal powers and the functions E, S, C, can be expressed in a closed form by means of known functions. Such a case will be discussed in the next section.

4. The case $\tau_1=\tau_2=y^{-p}$. In this section we shall consider the special case

$$\Sigma = \left\| \begin{matrix} 1 & y^{-p} \\ 1 & y^{-p} \end{matrix} \right\|. \tag{4.1}$$

For the sake of simplicity we assume that

$$p \geqq 0,$$

and set

$$p = 2q + 1. \tag{4.2}$$

Along the real axis $\tau_1=\tau_2$ vanishes (except in the trivial case $p=0$). However the integrals defining $Y^{(n)}$ exist, and therefore so do the functions $Z^{(n)}$, E, C, S. The integrals defining $Y^{*(n)}$ converge only if $p<1$. In this case only can we define the functions $i\cdot Z^{(n)}$, $i\cdot E$, $i\cdot C$, and $i\cdot S$.

From (2.5) we obtain by a simple calculation

$$\begin{aligned} Y^{(2\nu+1)}(y) &= \frac{1\cdot 3\cdot 5 \cdots (2\nu+1)}{(p+1)(p+3)\cdots(p+2\nu+1)}\, y^{p+2\nu+1}, \\ Y^{(2\nu)}(y) &= \frac{1\cdot 3\cdot 5 \cdots (2\nu-1)}{(p+1)(p+3)\cdots(p+2\nu-1)}\, y^{2\nu}, \end{aligned} \tag{4.3}$$

so that, by (4.3) and (2.6),

$$\begin{aligned} Z^{(n)}(z) = n!\Gamma(q+1)\Bigg\{ &\sum_{\nu=0}^{[n/2]} \frac{(-1)^\nu}{\nu!(n-2\nu)!\Gamma(q+\nu+1)}\, x^{n-2\nu}\left(\frac{y}{2}\right)^{2\nu} \\ &+ iy^p \sum_{\nu=0}^{[(n-1)/2]} \frac{(-1)^\nu}{\nu!(n-2\nu-1)!\Gamma(q+\nu+1)}\, x^{n-2\nu-1}\left(\frac{y}{2}\right)^{2\nu+1}\Bigg\}. \end{aligned}$$

Simpler expressions for $Z^{(n)}$ will be found later.

Next we determine the Σ-monogenic exponential and trigonometric functions. For the function s defined by (3.5) we get

$$
\begin{aligned}
s(\alpha, y) &= \sum_{\nu=0}^{\infty} \frac{(-1)^\nu 1\cdot 3 \cdots (2\nu+1)}{(2\nu+1)!(p+1)\cdots(p+2\nu+1)} \alpha^{2\nu+1} y^{p+2\nu+1} \\
&= \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \Gamma(q+1)}{2^\nu \nu! \Gamma(q+\nu+2) 2^{\nu+1}} \alpha^{2\nu+1} y^{p+2\nu+1} \\
&= \tfrac{1}{2}\Gamma(q+1)\alpha y^{p+1} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \Gamma(q)}{2^{2\nu}\nu!\Gamma(q+\nu+2)} (\alpha y)^{2\nu}.
\end{aligned}
$$

Recalling the definition of a Bessel function[3] we see that

$$s(\alpha, y) = 2^q\Gamma(q+1)\alpha^{-q}y^{q+1}J_{q+1}(\alpha y). \tag{4.4}$$

Similarly we prove that

$$c(\alpha, y) = 2^q\Gamma(q+1)\alpha^{-q}y^{-q}J_q(\alpha y). \tag{4.5}$$

For $p=0$ (i.e. $q=-\frac{1}{2}$) these formulae transform themselves into the familiar expressions for $I_{1/2}$ and $I_{-1/2}$. For in this case $c(\alpha, y)=\cos \alpha y$, $s(\alpha, y)=\sin \alpha y$.

By (3.4), (3.6), (4.4) and (4.5) we have

$$E(\alpha, z) = 2^q\Gamma(q+1)\alpha^{-q}e^{\alpha z}\{y^{-q}J_q(\alpha y) + iy^{q+1}J_{q+1}(\alpha y)\}, \tag{4.6}$$

$$S(\alpha, z) = 2^q\Gamma(q+1)(i\alpha)^{-q}\{y^{-q}\sin \alpha x J_q(i\alpha y) + y^{q+1}\cos \alpha x J_{q+1}(i\alpha y)\}, \tag{4.7}$$

$$C(\alpha, z) = 2^q\Gamma(q+1)(i\alpha)^{-q}\{y^{-q}\cos \alpha x J_q(i\alpha y) - y^{q+1}\sin \alpha x J_{q+1}(i\alpha y)\}. \tag{4.8}$$

In the case when

$$p > 1$$

the correspondence (at the origin) between analytic and Σ-monogenic functions can be expressed by an integral formula. Since only the $Z^{(n)}$'s are defined we may consider only such analytic functions which are real on the real axis.

For our case the equations (1.2) have the form

$$y\Delta u + pu_y = 0, \tag{4.9}$$

$$y\Delta v - pv_y = 0. \tag{4.10}$$

It is known that the solution of (4.9) which coincides with the analytic function $\varphi(z)$ on the real axis is given by the generalized Laplace integral[4]

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[3] See for instance, E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 3d ed., Cambridge Univ. Press, 1920, p. 359.

[4] See H. Bateman, *Partial differential equations of mathematical physics*, Cambridge University Press, 1932, p. 408.

$$u = \frac{1}{\pi}\int_0^\pi \varphi(x + iy\cos\alpha)\sin^{p-1}\alpha\, d\alpha.$$

A simple argument shows that a solution of (4.10), connected with u by (1.1), and vanishing on the real axis is given by

$$v = -iy^p\frac{1}{\pi}\int_0^\pi \varphi(x + iy\cos\alpha)\sin^{p-1}\alpha\cos\alpha\, d\alpha.$$

Thus the Σ-monogenic function corresponding (at the origin) to $\varphi(z)$, is given by

$$f(z) = \frac{1}{\pi}\int_0^\pi \varphi(x + iy\cos\alpha)(1 + y^p\cos\alpha)\sin^{p-1}\alpha\, d\alpha. \tag{4.11}$$

In particular we have

$$Z^{(n)}(z) = \frac{1}{\pi}\int_0^\pi (x + iy\cos\alpha)^n(1 + y^p\cos\alpha)\sin^{p-1}\alpha\, d\alpha. \tag{4.12}$$

The integral (4.11) represents a Σ-monogenic function even when φ possesses a pole at $z=0$. We define

$$Z^{(-n)}(z) = \frac{1}{\pi}\int_0^\pi (x + iy\cos\alpha)^{-n}(1 + y^p\cos\alpha)\sin^{p-1}\alpha\, d\alpha. \tag{4.13}$$

It is easily seen that

$$\frac{d_\Sigma}{d_\Sigma z}Z^{(-n)}(z) = -nZ^{(-n-1)}(z).$$

We now assume that p is an odd integer. In this case the formal powers can be expressed by Legendre polynomials.

Introducing polar coordinates r, φ,

$$x = r\cos\varphi, \qquad y = r\sin\varphi$$

and recalling that the associated Legendre functions P_n^m admit the representations[5]

$$r^n y^{-m} P_n^m(\cos\varphi) = \frac{1}{\pi}\,\frac{2^m\Gamma(m+1)}{\Gamma(2m+1)}\,\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}\int_0^\pi (x+iy\cos\alpha)^{n-m}\sin^{2m}\alpha\, d\alpha$$

we obtain from (4.12)

$$\operatorname{Re} Z^{(n)}(z) = \frac{\Gamma(p)\Gamma(n+1)}{2^q\Gamma(q+1)\Gamma(p+n)}\, r^n \sin^{-q}\varphi\, P_{n+q}^q(\cos\varphi). \tag{4.14}$$

[5] Bateman, l. c., p. 407.

Note that this formula is valid for an arbitrary $p>1$, if P_n^m, where m is not an integer, is defined by Hobson's formula.[6]

If, as we assumed, q is an integer, we have by definition that

$$P_{n+q}^{q}(\cos\varphi) = \sin^{q}\varphi\, P_{n+q}^{(q)}(\cos\varphi)$$

where parenthesis on the superscript indicates differentiation. Then, by (4.13),

$$\operatorname{Re} Z^{(n)}(z) = \frac{(p-1)!n!}{2^q q!(2q+n)!} r^n P_{n+q}^{(q)}(\cos\varphi).$$

By (2.2)

$$\operatorname{Im} Z^{(n)} = -\frac{y^p}{n+1}\frac{\partial}{\partial y}\operatorname{Re} Z^{(n+1)},$$

so that by (2.6)

$$\operatorname{Im} Z^{(n)} = \frac{(p-1)!n!}{2^q q!(p+n)!} y^p r^{n-1}\left\{\cos\varphi\, P_{n+q+1}^{(q+1)}(\cos\varphi) - (1+n)P_{n+q+1}^{(q)}(\cos\varphi)\right\}$$

and

$$\begin{aligned} Z^{(n)}(z) = \frac{(p-1)!n!}{2^q q!(p+n)!} r^n \Big\{ &(p+n)P_{n+q}^{(q)}(\cos\varphi) \\ &+ iy^p\sin\varphi\left[\cos\varphi P_{n+q+1}^{(q+1)}(\cos\varphi) - (n+1)P_{n+q+1}^{(q)}(\cos\varphi)\right]\Big\}. \end{aligned}$$

An analogous formula can be obtained for $Z^{(-n)}$.

If Σ has the form (4.1), particular solutions can be obtained also in another way. Introducing the polar coordinates r, φ as new independent variables, we obtain from (1.1) the system

$$\begin{cases} r^{p+1}u_r = \sin^{-p}\varphi\, v_\varphi \\ r^{p-1}u_\varphi = -\sin^{-p}\varphi\, v_r. \end{cases}$$

Denoting $r+i\varphi$ by ζ and setting

$$\Sigma_* = \left\| \begin{matrix} r^{p+1} & \sin^{-p}\varphi \\ r^{p-1} & \sin^{-p}\varphi \end{matrix} \right\|, \tag{4.15}$$

we see that a Σ-monogenic function of z is a Σ_*-monogenic function of ζ and vice versa.

Constructing Σ_*-monogenic formal powers, $Z_*^{(n)}(\zeta)$, we obtain particular Σ-monogenic functions of z.

For instance, if $p=0$ (Cauchy-Riemann equations), we have

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[6] Whittaker and Watson, loc. cit., p. 325.

$$\Sigma_* = \left\| \begin{matrix} r & 1 \\ \dfrac{1}{r} & 1 \end{matrix} \right\|$$

and

$$Z_* = \log z, \qquad i \cdot Z_* = i \log z. \tag{4.16}$$

Of course, this method works not only in the case (4.1) but also when τ_1 and τ_2 are of the form $x^k y^l (x^2+y^2)^m$.

5. Potential flow of an incompressible fluid with rotational symmetry. In this and the following sections we shall briefly consider some mechanical examples leading to equations of the form (1.1).

Perhaps the simplest example is that of a rotationally symmetric potential flow of an incompressible perfect fluid.[7] We introduce cylindrical co-ordinates ρ, θ, z, $\rho=0$ being the axis of symmetry. Let q_1, q_2, q_3 be the components of the velocity vector $\vec{q}$ in the direction of increasing ρ, θ, z. Then $q_2=0$ and $\partial q_1/\partial\theta=0$, $\partial q_3/\partial\theta=0$. Using the well known formulae for div $\vec{q}$ and curl $\vec{q}$ we may write the continuity equation in the form

$$\operatorname{div} \vec{q} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho q_1) + \frac{\partial q_3}{\partial z} = 0.$$

The condition of irrotationality takes the form

$$\left| \operatorname{curl} \vec{q} \right| = \left| \frac{\partial q_1}{\partial z} - \frac{\partial q_3}{\partial \rho} \right| = 0.$$

These equations imply the existence of a function u (velocity potential) and of a function v (Stokes' stream function) such that

$$\begin{aligned} q_1 &= u_\rho, & q_3 &= u_z, \\ \rho q_1 &= -v_z, & \rho q_3 &= v_\rho. \end{aligned} \tag{5.1}$$

$u+iv$ may be called the complex potential of the flow. We have, by (5.1)

$$\begin{aligned} u_z &= \frac{1}{\rho} v_\rho \\ u_\rho &= -\frac{1}{\rho} v_z. \end{aligned} \tag{5.2}$$

In order to use our previous notations we shall write x, y instead of z, ρ. Then we see that the complex potential $u+iv$ is a Σ-monogenic function of $x+iy=z$, where

[7] See, for instance, M. M. Munk, *Fluid mechanics*, Part II, in W. F. Durand, *Aerodynamic theory*, Springer, 1934, vol. 1, pp. 260–263.

$$\Sigma = \begin{Vmatrix} 1 & y^{-1} \\ 1 & y^{-1} \end{Vmatrix}.$$

Since Σ is of the form (4.1) we may use the results of section 4. For the formal powers we obtain [cf. (4.2) and (4.14)], when $q=0$, $x=r\cos\varphi$, $y=r\sin\varphi$,

$$Z^{(n)} = \frac{r^n}{n+1}\left\{(n+1)P_n(\cos\varphi) + iy\sin\varphi\left[\cos\varphi\, P'_{n+1}(\cos\varphi) - (n+1)P_{n+1}(\cos\varphi)\right]\right\}.$$

Using a well known identity this may be written as

$$Z^{(n)} = r^n P_n(\cos\varphi) + \frac{i}{n+1} r^{n-1} y^2 P'_n(\cos\varphi). \tag{5.3}$$

The real and the imaginary parts of the right hand side of (5.3) are the known polynomial solutions of the second order equations for u and v:

$$\begin{aligned} y\Delta u + u_y &= 0, \\ y\Delta v - v_y &= 0. \end{aligned} \tag{5.4}$$

Now we introduce in the x, y-plane polar co-ordinates r, φ (this amounts to introducing spherical co-ordinates in the physical space). Then $u+iv$ will be a Σ_*-monogenic function of $\zeta=r+i\varphi$, where [cf. (4.15)]

$$\Sigma_* = \begin{Vmatrix} r^2 & \sin^{-1}\varphi \\ 1 & \sin^{-1}\varphi \end{Vmatrix}.$$

Forming the first Σ_*-monogenic formal powers we get

$$\underset{1}{Z_*^{(1)}} = 1 - \frac{1}{r} + i(1-\cos\varphi) = -\left(\frac{1}{r} + i\cos\varphi\right) + \text{const.} \tag{5.5}$$

and

$$i\cdot \underset{i\pi/2}{Z_*^{(1)}} = -\log\tan\tfrac{1}{2}\varphi + ir. \tag{5.6}$$

In the case of a two-dimensional flow (Cauchy-Riemann equations) $Z_*^{(1)}$ was the complex potential of a source and $i\cdot Z_*^{(1)}$ that of a vortex [cf. (4.16)]. Similarly, in our case, (5.5) is recognized as the complex potential of a sink. (5.6) is the potential of a flow for which the streamlines are circles around the origin. The line $y=0$ is a singular line; it is covered by continuously distributed sources and sinks.

Other particular solutions can be obtained by forming higher Σ_*-monogenic powers and by iterated Σ-integration of (5.5) and (5.6).

Σ-differentiating (5.5) we obtain (exactly as in the case of a plane flow) the complex potential of a doublet. In fact this Σ-derivative equals

$$\frac{\cos\varphi}{r^2} - i\,\frac{\sin^2\varphi}{r}\cdot$$

Repeated Σ-differentiation leads to doublets of higher order. It is easily seen that the Σ-monogenic functions obtained in this way coincide with the formal negative power defined in section 4. In our case, ($p=1$), these powers have the form

$$Z^{(-n)} = r^{-n-1}P_n(\cos\varphi) - iy^2r^{-n-2}P_n'(\cos\varphi).$$

Σ-differentiation of $i\cdot Z_*$ does not lead to new particular solutions, since the Σ-derivative of (5.6) coincides with (5.5) (but for an additive constant).

Finally we note the form of the Σ-monogenic exponential and trigonometric functions. From (4.2) and (4.6)–(4.8) we obtain, setting $p=1$,

$$\begin{aligned} E(\alpha, z) &= e^{\alpha x}\{J_0(\alpha y) + iyJ_1(\alpha y)\} \\ S(\alpha, z) &= \sin\alpha x J_0(i\alpha y) + y\cos\alpha x J_1(i\alpha y) \\ C(\alpha, z) &= \cos\alpha x J_0(i\alpha y) - y\sin\alpha x J_1(i\alpha y). \end{aligned} \tag{5.7}$$

6. Torsion of elastic bodies of revolution.[8] We consider an elastic body of revolution and introduce cylindrical coordinates ρ, θ, z, $\rho=0$ being the axis of symmetry. The physical components of the stress tensor (in the above coordinates) shall be denoted by σ_{ik}, those of the displacement vector by u_i, $i, k=1, 2, 3$. Because of the symmetry these quantities do not depend upon θ. Furthermore, we assume that

$$u_1 = u_3 = 0. \tag{6.1}$$

Then all σ_{ik} vanish except σ_{12} and σ_{23}. The condition of equilibrium takes the form

$$\frac{\partial\sigma_{12}}{\partial\rho} + \frac{\partial\sigma_{23}}{\partial z} + \frac{2\sigma_{12}}{\rho} = 0 \tag{6.2}$$

whereas the stress-strain connection is given by

$$\sigma_{12} = G\frac{\partial}{\partial\rho}\left(\frac{u_2}{\rho}\right), \qquad \sigma_{23} = G\frac{\partial u_2}{\partial z}\cdot \tag{6.3}$$

[8] See A. E. H. Love, *A treatise on the mathematical theory of elasticity*, 4th ed., Cambridge University Press, 1934, pp. 330–332, and A. Timpe, *Die Torsion von Umdrehungskörpern*, Math. Annalen **71**, 480–509 (1912).

Substituting from (6.3) into (6.2) we obtain

$$\frac{\partial^2 u_3}{\partial \rho^2} + \frac{\partial^2 u_2}{\partial z^2} + \frac{1}{\rho}\,\frac{\partial u_2}{\rho} - \frac{u}{\rho^2} = 0. \tag{6.4}$$

If the function ψ is defined by

$$\psi = \rho u,$$

(6.4) may be written as

$$\psi_{\rho\rho} + \psi_{zz} - \frac{1}{\rho}\psi_\rho = 0$$

This equation implies the existence of a function φ connected with ψ by the equations

$$\begin{aligned} \varphi_z &= \frac{1}{\rho}\psi_\rho \\ \varphi_\rho &= -\frac{1}{\rho}\psi_z. \end{aligned} \tag{6.5}$$

φ and ψ can be interpreted as the velocity potential and Stokes' stream function of a rotationally symmetric potential flow [cf. (5.2)]. The particular solutions of (6.5) have been discussed in the preceeding section.

Let our body be a cylinder subjected to a deformation of the kind considered above. If the displacements on the boundary are given, the displacements in the interior can be found by integrating (6.4) under the boundary condition:

$$\psi = \chi(z), \quad \text{for} \quad \rho = \pm \rho_0,$$

ρ_0 being a constant and χ a given function. Plainly, this boundary value problem can be solved by representing χ by a Fourier series (or by a Fourier integral) and by using the particular solutions (5.7). Similarly the solutions (5.7) can be used if the boundary values are prescribed on the lines $z=$const. In this case developments in series of Bessel functions should be used. The discussion of convergence will be found in the quoted paper of A. Timpe.

Equation (6.2) can also be written in the form

$$\frac{\partial}{\partial \rho}(\rho^2\sigma_{12}) + \frac{\partial}{\partial z}(\rho^2\sigma_{23}) = 0.$$

This equation can be satisfied by introducing a stress function v and setting

$$\rho^2\sigma_{12} = -\frac{\partial v}{\partial z}, \qquad \rho^2\sigma_{23} = \frac{\partial v}{\partial \rho} \tag{6.6}$$

On the other hand, writing

$$u = \frac{Gu_2}{\rho},$$

we obtain from (6.3)

$$\sigma_{12} = \rho \frac{\partial u}{\partial \rho}, \qquad \sigma_{23} = \rho \frac{\partial u}{\partial z}. \tag{6.7}$$

Comparing (6.6) and (6.7) we get

$$\begin{aligned} u_z &= \frac{1}{\rho^3} v_\rho \\ u_\rho &= -\frac{1}{\rho^3} v_z. \end{aligned} \tag{6.8}$$

The *torsion problem* consists of integrating equations (6.8) under the boundary condition

$$v = \text{const. on the boundary.}$$

This condition expresses the fact that the surface of revolution is free of stresses.

In order to maintain our previous notations we shall write x, y instead of z, ρ. Then (6.8) shows that $u+iv$ is a Σ-monogenic function of $x+iy=z$, where

$$\Sigma = \left\| \begin{matrix} 1 & y^{-3} \\ 1 & y^{-3} \end{matrix} \right\|.$$

The second order equations for u and v take the form

$$y\Delta u + 3u_y = 0, \tag{6.9}$$

$$y\Delta v - 3v_y = 0. \tag{6.10}$$

Again Σ has the particular form discussed in section 4. From (4.2) and (4.14) we obtain, setting $p=3$,

$$\begin{aligned} Z^{(n)}(z) = \frac{n!}{(n+3)!} r^n \{ &(n+3)P'_{n+1}(\cos\varphi) \\ &+ iy^3 \sin\varphi[\cos\varphi\, P''_{n+2}(\cos\varphi) - (n+1)P'_{n+2}(\cos\varphi)]\} \end{aligned}$$

(r, φ are polar coordinates). This is easily transformed into

$$Z^{(n)} = \frac{1}{(n+1)(n+2)} \{ r^n P'_{n+1}(\cos\varphi) + ir^{n+3} \sin^4\varphi\, P''_{n+1}(\cos\varphi)\}, \tag{6.11}$$

$$n = 1, 2, \cdots.$$

The imaginary parts of (6.11) are the known polynomial solutions of (6.10).[9]

Next we form the Σ-monogenic exponential and trigonometric functions. From (4.6)–(4.8) we obtain

$$E(\alpha, z) = \frac{2}{\alpha} e^{\alpha x}\left\{\frac{1}{y} J_1(\alpha y) + iy^2 J_2(\alpha y)\right\}$$

$$S(\alpha, z) = \frac{2}{\alpha}\left\{\frac{1}{y} \sin \alpha x J_1(i\alpha y) + y^2 \cos \alpha x J_2(i\alpha y)\right\}$$

$$C(\alpha, z) = \frac{2}{\alpha}\left\{\frac{1}{y} \cos \alpha x J_1(i\alpha y) - y^2 \sin \alpha x J_2(i\alpha y)\right\}.$$

The imaginary part of E again coincides with a well known particular solution of (6.10).[10]

If we introduce the polar coordinates as new independent variables, $u+iv$ becomes a Σ_*-monogenic function of $\zeta = r+i\varphi$, where

$$\Sigma_* = \left\| \begin{matrix} r^4 & \sin^{-3}\varphi \\ r^2 & \sin^{-3}\varphi \end{matrix} \right\|.$$

Forming the first Σ_*-monogenic formal powers we obtain

$$Z_*^{(1)} = 1 - \frac{1}{3r^3} + i\left[\tfrac{2}{3} - \tfrac{1}{3}\cos\varphi\,(\sin^2\varphi + 2)\right] \tag{6.12}$$

$$i\cdot \underset{i\pi/2}{Z_*^{(1)}} = \frac{\cos\varphi}{2\sin^2\varphi} - \tfrac{1}{2}\log\tan\tfrac{1}{2}\varphi + i\frac{r^3}{3}. \tag{6.13}$$

(6.12) describes the torsion of a cone with the vertex at the origin. (6.13) represents the torsion of a sphere around the origin, the torque being transmitted through a singular line along a diameter.

Σ-differentiating (6.12) we get

$$\frac{d_\Sigma}{d_\Sigma z} Z_*^{(1)} = \frac{\cos\varphi}{r^4} - i\frac{\sin^4\varphi}{r}. \tag{6.14}$$

The equations (6.8) can also be interpreted hydrodynamically. $u-iv$ can be understood to be a complex potential [(velocity potential)$+i$(Stokes' stream function)] of a rotationally symmetric potential flow in a five dimensional Euclidean space. (This interpretation is due to Arndt).[11] If we adopt this point of view, $Z_*^{(1)}$ again represents a source, and its Σ-derivative, (6.14),

[9] Love, l. c., p. 331.

[10] Love, l. c., p. 332.

[11] F. C. Arndt, *Die Torsion von Wellen mit achsensymmetrischen Bohrungen und Hohlräumen*, Thesis, Göttingen, 1916.

a doublet. Combining "flows" of the form (6.12) and (6.14) Arndt solved the torsion problem for a number of bodies with holes.

Repeated Σ-differentiation of (6.14) leads to doublets of higher order. These complex potentials are equivalent to the $Z^{(-n)}$'s defined in section 4.

7. Two dimensional potential gas flow. Let p and ρ represent the pressure and density of a perfect compressible fluid.[12] If heat conduction is neglected the above quantities are connected by the relation

$$p = (p_0/\rho_0^\gamma)\rho^\gamma,$$

where γ is the ratio of the specific heat for constant pressure to the specific heat for constant volume, the subscript zero referring to the fluid at rest. Introducing the local velocity of sound, a, given by

$$a^2 = \frac{dp}{d\rho} = \gamma \frac{p_0}{\rho_0^\gamma} \rho^{\gamma-1},$$

we have by Bernoulli's equation

$$a^2 = a_0^2 - [(\gamma - 1)/2]q^2,$$

where q is the magnitude of the velocity, and

$$\rho = \rho_0\left[1 - \frac{\gamma - 1}{2} \frac{q^2}{a_0^2}\right]^{1/(\gamma-1)}.$$

The continuity equation for a steady flow has the form

$$\operatorname{div}\left(\frac{\rho}{\rho_0}\vec{q}\right) = 0, \tag{7.1}$$

where $\vec{q}$ is the velocity vector. If the flow is irrotational,

$$\operatorname{curl} \vec{q} = 0. \tag{7.2}$$

For a two-dimensional flow, $\vec{q} = (q\cos\theta, q\sin\theta, 0)$, θ giving the direction of the flow. (7.1) and (7.2) then imply the existence of two functions, $\Phi(x, y)$, $\Psi(x, y)$, such that

$$q\cos\theta = \Phi_x \qquad q\sin\theta = \Phi_y,$$

and

$$q\cos\theta = \frac{\rho_0}{\rho}\Psi_y, \qquad q\sin\theta = -\frac{\rho_0}{\rho}\Psi_x.$$

[12] See, for instance, G. I. Taylor and J. W. MacColl, *The mechanics of compressible fluids*, in W. F. Durand, *Aerodynamic theory*, Springer, 1935, vol. 3, pp. 229–230.

Thus the velocity potential Φ and the stream function Ψ are connected by the equations

$$\Phi_x = \frac{\rho_0}{\rho} \Psi_y$$

$$\Phi_y = -\frac{\rho_0}{\rho} \Psi_x.$$

Since ρ is a non-linear function of the derivatives of the unknown functions Φ and Ψ, the above equations are non-linear. However, linear equations can be obtained by a transformation due to Molenbroek[13] and to Chaplygin;[14] namely, by introducing as new independent variables the quantities θ and q. The equations then become

$$\begin{aligned} \Phi_\theta &= \frac{\rho_0}{\rho} q \Psi_q \\ \Phi_q &= -\frac{\rho_0}{\rho}\left(1 - \frac{q^2}{a^2}\right)\frac{1}{q}\Psi_\theta. \end{aligned} \tag{7.3}$$

To simplify the formulae Chaplygin introduces a new variable

$$t = \frac{2(\gamma - 1)}{\gamma \rho_0^{\gamma-1}} q^2.$$

In what follows we write x, y instead of θ, t. (x, y are *not*, as they were before, the coordinates in the physical plane). Then (7.3) shows that the complex potential $\Phi + i\Psi$ is a Σ-monogenic function of $x + iy$, with

$$\Sigma = \left\| \begin{matrix} 1 & \dfrac{2y}{(1-y)^m} \\ 1 & \dfrac{1-(2m+1)y}{2y(1-y)^{1+m}} \end{matrix} \right\|, \qquad m = \frac{1}{\gamma - 1}.$$

We see that

$$\tau_1 = 1/\tau_2 = 0 \quad \text{for} \quad y = 0, \qquad \tau_2 = 0 \quad \text{for} \quad y = 1/(2m+1). \tag{7.4}$$

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[13] P. Molenbroek, *Über einige Bewegungen eines Gases mit Annahme eines Geschwindigkeitspotentials*, Arch. d. Mathem. u. Phys. (2). **9**, 157 (1890).

[14] A. Chaplygin, *On gas jets*, Scientific Annals of the Moscow University, Section for Math. and Phys. **21**, 1–121 (1904) (Russian). Cf. Th. von Kármán, *Compressibility effects in aerodynamics*, Journ. Aeronaut. Sc. **8**, 337–356 (1941). (This paper also contains an extensive bibliography.) For a detailed description of the Chaplygin transformation in English see S. Bergman, *The Hodograph Method in the Theory of Compressible Fluids* (suppl. to K. O. Friedrichs, R. v. Mises, *Fluid Dynamics*), mimeographed lecture notes, Brown University, 1942.

The formal powers $a \cdot Z_{z_0}^{(n)}$ cannot be defined for $\operatorname{Im} z_0 = 0$. For

$$0 < y < 1/(2m+1), \tag{7.5}$$

i.e. for *subsonic flow*, we have the elliptic case, $\tau_i > 0$, $i = 1, 2$.

We postpone a systematic discussion of the application of our method to this case and shall show only how an important set of particular solutions (due to Chaplygin) can be represented in terms of our functions.

The second order equation for Ψ takes the form

$$\frac{1-(2m+1)y}{2y(1-y)^{1+m}}\Psi_{xx} + \frac{\partial}{\partial y}\left\{2y(1-y)^{-m}\Psi_y\right\} = 0. \tag{7.6}$$

Chaplygin gives as particular solutions of (7.6)[15]

$$\psi = y^\alpha F_\alpha(y) \cos \alpha x,$$
$$\chi = y^\alpha F_\alpha(y) \sin \alpha x,$$

where $\alpha > 0$ and $F_\alpha(y)$ is the hypergeometric function

$$F_\alpha(y) = F(a_\alpha, b_\alpha, 2\alpha + 1, y)$$

the constants a_α, b_α being determined by the conditions

$$a_\alpha + b_\alpha = 2\alpha - m, \qquad a_\alpha b_\alpha = -m\alpha(2\alpha + 1).$$

Consider the Σ-monogenic function

$$H = \varphi + i\psi;$$

we have

$$\begin{aligned} H^{\backprime} &= \varphi_x + i\psi_x = 2y(1-y)^{-m}\psi_y + i\psi_x \\ &= 2y(1-y)^{-m}y^{\alpha-1}\left[\alpha F_\alpha(y) + yF_\alpha'(y)\right]\cos \alpha x - i\alpha y^\alpha F_\alpha(y) \sin \alpha x. \end{aligned}$$

Therefore

$$h = H^{\backprime},$$

satisfies the differential equation

$$\frac{\partial^2 h}{\partial x^2} + \alpha^2 h = 0,$$

i.e. the Σ-differential equation

$$h^{\backprime\backprime} + \alpha^2 h = 0. \tag{7.7}$$

[15] Chaplygin, l. c. Similar solutions have been considered by F. Ringleb, *Exakte Lösungen der Differentialgleichungen einer adiabatischen Gasströmung*, Zeitschr. angew. Math. Mech. **20**, 185–198 (1940). The method by which Chaplygin solved the jet problem and Ringleb computed the complex potentials of a compressible doublet is actually identical with the use of the "correspondence" discussed in section 3.

Let y_0 be a fixed real number, satisfying (7.5). Then

$$h(iy_0) = \alpha A, \qquad h'(iy_0) = i\alpha^2 B \tag{7.8}$$

where

$$\begin{aligned} A &= \frac{1}{\alpha} 2y_0(1 - y_0)^m y_0^{\alpha-1}[\alpha F_\alpha(y_0) + y_0 F'_\alpha(y_0)] \\ B &= - y_0^\alpha F_\alpha(y_0). \end{aligned} \tag{7.9}$$

It is seen easily that the solution of (7.7) with the initial conditions (7.9) is

$$A \underset{iy_0}{C}(\alpha, z) + Bi \cdot \underset{iy_0}{S}(\alpha, z).$$

This function is therefore identical with h. Σ-integrating and suppressing a non-essential constant we obtain

$$H = A \underset{iy_0}{S}(\alpha, z) - Bi \cdot \underset{iy_0}{C}(\alpha, z)$$

so that

$$\psi = \operatorname{Im}[A \underset{iy_0}{S}(\alpha, z) - Bi \cdot \underset{iy_0}{C}(\alpha, z)].$$

In a similar way we obtain

$$\chi = - \operatorname{Im}[A \underset{iy_0}{C}(\alpha, z) + Bi \cdot \underset{iy_0}{S}(\alpha, z)].$$

It is interesting to note that if α is a positive integer, $H = \varphi + i\psi$ and $G = \omega + i\chi$ are regular at the origin and that all their Σ-derivatives vanish at this point. This is possible because of (7.4).

We also note that the first formal powers

$$Z^{(1)} \quad \text{and} \quad i \cdot Z^{(1)}$$

represent the complex potentials of a compressible source and vortex respectively.[16]

[16] Cf. Ringleb, l. c.