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AUSLANDER-REITEN COMPONENTS DETERMINED BY THEIR COMPOSITION FACTORS

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Abstract. We provide sufficient conditions for a component of the Auslander-Reiten quiver of an Artin algebra to be determined by the composition factors of its indecomposable modules.

1. Introduction and main results

Let A be an Artin algebra over a commutative Artin ring R. We denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, by $K_0(A)$ the Grothendieck group of A, and by $[M]$ the image of a module M from $\operatorname{mod} A$ in $K_0(A)$. Thus, for modules M and N in $\operatorname{mod} A$, $[M] = [N]$ if and only if M and N have the same composition factors including the multiplicities. An interesting open problem is to find handy criteria for two indecomposable modules M and N in $\operatorname{mod} A$ with the same composition factors to be isomorphic. It was shown in [16] that this is the case when M does not lie on a short cycle $M \to X \to M$ of non-zero non-isomorphisms in $\operatorname{mod} A$ with X an indecomposable module, generalizing earlier results about directing modules proved in [6], [8]. In fact, it follows from [7] and [16] that an indecomposable module M in $\operatorname{mod} A$ lies on a short cycle $M \to X \to M$ in $\operatorname{mod} A$ if and only if M is the middle term of a chain $Y \to M \to D\operatorname{Tr} Y$ of non-zero homomorphisms in $\operatorname{mod} A$ with Y a non-projective indecomposable module. Hence the above result from [16] gives in fact another interpretation of a result from [3]. An important combinatorial and homological invariant of the module category $\operatorname{mod} A$ of an Artin algebra A is its Auslander-Reiten quiver Γ_A [4]. Sometimes, we may recover the algebra A and the category $\operatorname{mod} A$ from the shape of the components $\mathscr{C}$ of Γ_A and their behaviour in the category $\operatorname{mod} A$. By a component of Γ_A we mean a connected component of the translation quiver Γ_A.

In this article we are concerned with the problem of finding handy criteria for a component $\mathscr{C}$ of the Auslander-Reiten quiver Γ_A of an Artin algebra A to be uniquely determined in Γ_A by the composition factors of its indecomposable modules. We say that two components $\mathscr{C}$ and $\mathscr{D}$ of Γ_A have the same composition factors if, for any element $\mathbf{x} \in K_0(A)$, $\mathbf{x} = [M]$ for an indecomposable module M in $\mathscr{C}$ if and only if $\mathbf{x} = [N]$ for an indecomposable module N in $\mathscr{D}$.

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In order to state the main results, we recall some concepts. For an Artin algebra A, we denote by rad_A the Jacobson radical of $\operatorname{mod} A$, generated by all nonisomorphisms between indecomposable modules in $\operatorname{mod} A$, and by $\operatorname{rad}_A^\infty$ the infinite radical of $\operatorname{mod} A$, which is the intersection of all powers rad_A^i, $i \geq 1$, of rad_A. Recall that, by a result of M. Auslander [2], $\operatorname{rad}_A^\infty = 0$ if and only if A is of finite representation type, that is, there are in $\operatorname{mod} A$ only finitely many indecomposable modules up to isomorphism. Following [24], a component quiver Σ_A of A is the quiver whose vertices are the components $\mathscr{C}$ of Γ_A, and two components $\mathscr{C}$ and $\mathscr{D}$ of Γ_A are linked in Σ_A by an arrow $\mathscr{C} \to \mathscr{D}$ provided $\operatorname{rad}_A^\infty(X, Y) \neq 0$ for some modules $X \in \mathscr{C}$ and $Y \in \mathscr{D}$. We note that a component $\mathscr{C}$ of Γ_A is generalized standard in the sense of [22] if and only if Σ_A has no loop at $\mathscr{C}$. By a short cycle in Σ_A we mean a cycle $\mathscr{C} \to \mathscr{D} \to \mathscr{C}$, where possibly $\mathscr{C} = \mathscr{D}$. We also mention that a component $\mathscr{C}$ of Γ_A lies on a short cycle $\mathscr{C} \to \mathscr{D} \to \mathscr{C}$ in Σ_A with $\mathscr{C} \neq \mathscr{D}$ if and only if $\mathscr{C}$ has an external short path $X \to Y \to Z$ with X and Z in $\mathscr{C}$ and Y in $\mathscr{D}$ [15]. Recall also that a translation quiver of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geq 1$, is called a stable tube of rank r. We note that every regular component (without projective modules and injective modules) of the Auslander-Reiten quiver Γ_A of an Artin algebra A is either a stable tube or is acyclic (without oriented cycles) of the form $\mathbb{Z}\Delta$ for an acyclic locally finite connected valued quiver Δ (see [13], [27]).

The following theorem is the first main result of this article.

Theorem 1. *Let A be an Artin algebra and $\mathscr{C}$ and $\mathscr{D}$ be two components of Γ_A with the same composition factors. Assume that $\mathscr{C}$ is not a stable tube of rank one and does not lie on a short cycle in Σ_A. Then $\mathscr{C} = \mathscr{D}$.*

The above theorem says that a generalized standard Auslander-Reiten component $\mathscr{C}$ of an Artin algebra A without external short paths, different from a stable tube of rank one, is uniquely determined in Γ_A by the composition factors of its indecomposable modules. We point out that the assumption on $\mathscr{C}$ not being a stable tube of rank one is essential for the validity of the above theorem. For example, if H is the path algebra $K\Delta$ of a Euclidean quiver Δ over an algebraically closed field K, then the component quiver Σ_H of H is acyclic and the Auslander-Reiten quiver Γ_H of H contains infinitely many pairwise different stable tubes of rank one having the same composition factors (see [17], [20]).

The second main result of the article clarifies the situation in general.

Theorem 2. *Let A be an Artin algebra, $\mathscr{C}$ a stable tube of rank one in Γ_A which does not lie on a short cycle in Σ_A, and $\mathscr{D}$ a component of Γ_A different from $\mathscr{C}$ and having the same composition factors as $\mathscr{C}$. Then there is a quotient algebra B of A such that the following statements hold:*

(a) *B is a concealed canonical algebra.*
(b) *$\mathscr{C}$ and $\mathscr{D}$ are stable tubes of a separating family of stable tubes of Γ_B.*
(c) *$\mathscr{D}$ is a stable tube of rank one.*

Recall that a concealed canonical algebra is an algebra of the form $B = \operatorname{End}_\Lambda(T)$, where Λ is a canonical algebra in the sense of C. M. Ringel [19] (see also [17]) and T is a multiplicity-free tilting module in the additive category $\operatorname{add}(\mathcal{P}^\Lambda)$, for the canonical decomposition $\Gamma_\Lambda = \mathcal{P}^\Lambda \vee \mathcal{T}^\Lambda \vee \mathcal{Q}^\Lambda$ of Γ_Λ, with $\mathcal{T}^\Lambda$ the canonical infinite separating family of stable tubes of Γ_Λ. Then Γ_B admits a decomposition $\Gamma_B = \mathcal{P}^B \vee \mathcal{T}^B \vee \mathcal{Q}^B$, where the image $\mathcal{T}^B = \operatorname{Hom}_\Lambda(T, \mathcal{T}^\Lambda)$ of the family $\mathcal{T}^\Lambda$ via the functor $\operatorname{Hom}_\Lambda(T, -) : \operatorname{mod} \Lambda \to \operatorname{mod} B$ is an infinite separating family of stable tubes of Γ_B. Moreover, all but finitely many stable tubes of $\mathcal{T}^B$ have rank one and the same composition factors. We also mention that, by a result of H. Lenzing and J. A. de la Peña [11], the class of concealed canonical algebras coincides with the class of Artin algebras whose Auslander-Reiten quiver admits a separating family of stable tubes.

We exhibit in Section 3 examples of generalized standard stable tubes of arbitrary large rank which are not uniquely determined by the composition factors. It would be interesting to clarify if an acyclic generalized standard regular component of the Auslander-Reiten quiver of an Artin algebra is uniquely determined by its composition factors (see Section 3 for related comments).

For basic background on the representation theory applied here we refer to [1], [4], [17], [20], [21].

2. Proofs of Theorems 1 and 2

Let A be an Artin algebra over a commutative Artin ring R. We denote by τ_A and τ_A^- the Auslander-Reiten translations $D\,\mathrm{Tr}$ and $\mathrm{Tr}\,D$, respectively. For a module V in $\mathrm{mod}\,R$, we denote by $|V|$ its length over R. In the proofs a crucial role will be played by the following formulas from [23, Proposition 4.1], their being consequences of [3, (1.4)] (see also [4, Corollary IV.4.3]).

For indecomposable modules M, N and X in $\mathrm{mod}\,A$ with $[M] = [N]$ the following equalities hold:

(i) $|\mathrm{Hom}_A(X, M)| - |\mathrm{Hom}_A(M, \tau_A X)| = |\mathrm{Hom}_A(X, N)| - |\mathrm{Hom}_A(N, \tau_A X)|$,

(ii) $|\mathrm{Hom}_A(M, X)| - |\mathrm{Hom}_A(\tau_A^- X, M)| = |\mathrm{Hom}_A(N, X)| - |\mathrm{Hom}_A(\tau_A^- X, N)|$.

Let $\mathscr{C}$ and $\mathscr{D}$ be components of Γ_A with the same composition factors such that $\mathscr{C}$ does not lie on a short cycle in Σ_A. We assume that $\mathscr{C} \neq \mathscr{D}$ and show in several steps that $\mathscr{C}$ and $\mathscr{D}$ are stable tubes of rank one of a separating family of stable tubes in the Auslander-Reiten quiver Γ_B of a concealed canonical algebra B.

(1) $\mathscr{C}$ is a semi-regular component of Γ_A ($\mathscr{C}$ does not contain both a projective module and an injective module). Assume $\mathscr{C}$ contains a projective module P and an injective module I. Since $\mathscr{C}$ and $\mathscr{D}$ have the same composition factors, there exist modules M and N in $\mathscr{D}$ such that $[P] = [M]$ and $[I] = [N]$. Then we have $\mathrm{Hom}_A(P, M) \neq 0$ and $\mathrm{Hom}_A(N, I) \neq 0$, because the top of P is a composition factor of M, and the socle of I is a composition factor of N. Hence, we have in Σ_A the short cycle $\mathscr{C} \to \mathscr{D} \to \mathscr{C}$, because $\mathrm{Hom}_A(P, M) = \mathrm{rad}_A^\infty(P, M)$ and $\mathrm{Hom}_A(N, I) = \mathrm{rad}_A^\infty(N, I)$, a contradiction. Therefore, $\mathscr{C}$ is a semi-regular component of Γ_A.

(2) $\mathscr{C}$ is a cyclic component of Γ_A (every module in $\mathscr{C}$ lies on an oriented cycle in $\mathscr{C}$). Take a module X in $\mathscr{C}$. It follows from our assumption that $[X] = [Y]$ for some module Y in $\mathscr{D}$, and so X is not uniquely determined by $[X]$, because $\mathscr{C} \neq \mathscr{D}$. Applying [16, Corollary 2.2], we conclude that we have in $\mathrm{mod}\,A$ a short cycle $X \to Z \to X$. Observe that then Z belongs to $\mathscr{C}$, because $\mathscr{C}$ does not lie on a short cycle in Σ_A. Moreover, since there is no loop at $\mathscr{C}$ in Σ_A, $\mathscr{C}$ is a generalized standard component of Γ_A, and hence $\mathrm{rad}_A^\infty(X, Z) = 0$ and $\mathrm{rad}_A^\infty(Z, X) = 0$. Then $\mathrm{Hom}_A(X, Z) \neq 0$ and $\mathrm{Hom}_A(Z, X) \neq 0$ imply that there exist paths of irreducible homomorphisms in $\mathrm{mod}\,A$ from X to Z and from Z to X (see [4, Proposition V.7.5]), and consequently there exists an oriented cycle in $\mathscr{C}$ passing through X and Z. Hence, $\mathscr{C}$ is a cyclic component.

(3) $\mathscr{C}$ is a ray tube (obtained from a stable tube by a finite number (possibly empty) of ray insertions) or a coray tube (obtained from a stable tube by a finite

number (possibly empty) of coray insertions) in the sense of [17, (4.5)] (see also [21, XV.2]). This is a direct consequence of [14, (2.6)], since by (1) and (2) $\mathscr{C}$ is semi-regular with oriented cycles.

(4) We may assume (without loss of generality) that $\mathscr{C}$ is a ray tube, hence without injective modules. Let $\operatorname{ann}_A(\mathscr{C})$ be the annihilator of $\mathscr{C}$ in A, that is, the intersection of the annihilators $\operatorname{ann}_A(X) = \{a \in A \mid Xa = 0\}$ of all modules X in $\mathscr{C}$, and $B = A/\operatorname{ann}_A(\mathscr{C})$. Then $\mathscr{C}$ is a faithful component of Γ_B. Since $\mathscr{C}$ does not lie on a short cycle in Σ_A, we conclude that $\mathscr{C}$ is without external short paths [15]; that is, there are no paths $U \to V \to W$ in $\operatorname{mod} A$ with U and W in $\mathscr{C}$ but V not in $\mathscr{C}$. Then it follows from [9, Theorem 2] that B is an almost concealed canonical algebra and $\mathscr{C}$ is a faithful ray tube of a separating family $\mathcal{T}^B$ of ray tubes of Γ_B. Recall that then there exists a canonical algebra Λ (in the sense of C. M. Ringel [17], [19]) such that $B = \operatorname{End}_\Lambda(T)$ for a tilting module T in the additive category $\operatorname{add}(\mathcal{P}^\Lambda \cup \mathcal{T}^\Lambda)$ of $\mathcal{P}^\Lambda \cup \mathcal{T}^\Lambda$, for the canonical decomposition $\Gamma_\Lambda = \mathcal{P}^\Lambda \vee \mathcal{T}^\Lambda \vee \mathcal{Q}^\Lambda$ of Γ_Λ with $\mathcal{T}^\Lambda$ the canonical separating family of stable tubes. By general theory (see [11], [12], [17], [19], [25]), Γ_B admits a decomposition

$$\Gamma_B = \mathcal{P}^B \vee \mathcal{T}^B \vee \mathcal{Q}^B,$$

where $\mathcal{T}^B$ is a family of ray tubes separating $\mathcal{P}^B$ from $\mathcal{Q}^B$ (in the sense of [19]). In particular, $\mathcal{T}^B$ is an infinite family of pairwise orthogonal generalized standard ray tubes, $\operatorname{Hom}_B(\mathcal{T}^B, \mathcal{P}^B) = 0$, $\operatorname{Hom}_B(\mathcal{Q}^B, \mathcal{T}^B) = 0$, and $\operatorname{Hom}_B(\mathcal{Q}^B, \mathcal{P}^B) = 0$. In fact, since $\mathscr{C}$ is a faithful ray tube of $\mathcal{T}^B$, all ray tubes of $\mathcal{T}^B$ except $\mathscr{C}$ are stable tubes. Moreover, the separation property of $\mathcal{T}^B$ implies that $\operatorname{Hom}_B(\mathscr{X}, \mathscr{C}) \neq 0$ for any component $\mathscr{X}$ from $\mathcal{P}^B$ and $\operatorname{Hom}_B(\mathscr{C}, \mathscr{Y}) \neq 0$ for any component $\mathscr{Y}$ from $\mathcal{Q}^B$. Moreover, we note that $\mathcal{Q}^B$ contains all indecomposable injective B-modules.

(5) $\mathscr{D}$ is a component of Γ_B. Write $A = P' \oplus P''$, where the simple summands of $P'/\operatorname{rad} P'$ are exactly the simple composition factors of modules in $\mathscr{C}$. Denote by $t_{P''}(A)$ the ideal of A generated by the images of all homomorphisms in $\operatorname{mod} A$ from P'' to A. Since $\mathscr{C}$ is a semi-regular component of Γ_A without external short paths, it follows from arguments in [15, Section 1] that $\operatorname{End}_A(P') \cong A/t_{P''}(A)$ and $t_{P''}(A) = \operatorname{ann}_A(\mathscr{C})$. Observe that $1_A = e + f$ for orthogonal idempotents e and f in A with $P' = eA$ and $P'' = fA$, and consequently $\operatorname{End}_A(P') \cong eAe$ and $t_{P''}(A) = AfA$. Clearly, then $B = A/\operatorname{ann}_A(\mathscr{C}) \cong eAe$. On the other hand, since $\mathscr{D}$ has the same composition factors as $\mathscr{C}$, we have $Nf = \operatorname{Hom}_A(fA, N) = \operatorname{Hom}_A(P'', N) = 0$, and consequently $N \operatorname{ann}_A(\mathscr{C}) = N(AfA) = (Nf)A = 0$, for any module N in $\mathscr{D}$. This shows that $\mathscr{D}$ is a component of Γ_B.

(6) $\mathscr{D}$ is a component of $\mathcal{T}^B$. Assume $\mathscr{D} \notin \mathcal{T}^B$. Fix a stable tube $\mathcal{T}^*$ of $\mathcal{T}^B$ of rank one, which is different from $\mathscr{C}$. By general theory ([11], [12], [19]), B is a tubular (branch) extension of a concealed canonical algebra C such that $\Gamma_C = \mathcal{P}^C \vee \mathcal{T}^C \vee \mathcal{Q}^C$, where $\mathcal{T}^C$ is a separating family of stable tubes, $\mathcal{P}^B = \mathcal{P}^C$, $\mathscr{C}$ is obtained from a stable tube $\mathcal{T}$ of $\mathcal{T}^C$ by a finite number (possibly empty) of ray insertions and the remaining tubes of $\mathcal{T}^C$ and $\mathcal{T}^B$ coincide ($\mathcal{T}^C \setminus \mathcal{T} = \mathcal{T}^B \setminus \mathscr{C}$). Clearly, C is a quotient algebra of B.

Let M be a module in $\mathscr{C}$ which lies in $\mathcal{T}$. In particular, the composition factors of M are C-modules. Take a module $N \in \mathscr{D}$ such that $[M] = [N]$. Assume $\mathscr{D} \in \mathcal{Q}^B$. Since $[M] = [N]$ there exists a projective module $P \in \mathcal{P}^B = \mathcal{P}^C$ such that $\operatorname{Hom}_B(P, N) \neq 0$. By the separation property of $\mathcal{T}^B$ we have $\operatorname{Hom}_B(X, N) \neq 0$

for some module $X \in \mathcal{T}^*$. Then, applying formula (i), we obtain

$$\begin{aligned} 0 = |\mathrm{Hom}_A(X, M)| - |\mathrm{Hom}_A(M, \tau_A X)| &= |\mathrm{Hom}_A(X, N)| - |\mathrm{Hom}_A(N, \tau_A X)| \\ &= |\mathrm{Hom}_A(X, N)| > 0, \end{aligned}$$

since M and X belong to orthogonal tubes of $\mathcal{T}^B$ and $\mathrm{Hom}_B(\mathcal{Q}^B, \mathcal{T}^B) = 0$. Dually, if $\mathscr{D} \in \mathcal{P}^B$, then there exists an injective module I in $\mathcal{Q}^B$ such that $\mathrm{Hom}_B(N, I) \neq 0$. By the separation property of $\mathcal{T}^B$, we have $\mathrm{Hom}_B(N, Y) \neq 0$ for some module $Y \in \mathcal{T}^*$. Then, by formula (ii), we have

$$\begin{aligned} 0 = |\mathrm{Hom}_A(M, Y)| - |\mathrm{Hom}_A(\tau_A^- Y, M)| &= |\mathrm{Hom}_A(N, Y)| - |\mathrm{Hom}_A(\tau_A^- Y, N)| \\ &= |\mathrm{Hom}_A(N, Y)| > 0, \end{aligned}$$

since M and Y belong to orthogonal tubes of $\mathcal{T}^B$ and $\mathrm{Hom}_B(\mathcal{T}^B, \mathcal{P}^B) = 0$. The above contradictions show that $\mathscr{D} \in \mathcal{T}^B$.

(7) $\mathcal{T}^B$ is a family of stable tubes. Assume $\mathscr{C}$ contains a projective module P. Take an indecomposable module Y in $\mathscr{D}$ with $[P] = [Y]$. Then the top of P is a composition factor of Y and hence $\mathrm{Hom}_B(P, Y) \neq 0$. Therefore, $\mathrm{Hom}_B(\mathscr{C}, \mathscr{D}) \neq 0$, which contradicts the fact that $\mathscr{C}$ and $\mathscr{D}$ are orthogonal. We conclude that $\mathscr{C}$ is a stable tube of $\mathcal{T}^B$. Clearly, then $\mathcal{T}^B$ is a separating family of stable tubes of Γ_B, and consequently B is a concealed canonical algebra, by [11].

(8) $\mathscr{C}$ and $\mathscr{D}$ are stable tubes of rank one. Since $\mathscr{C}$ and $\mathscr{D}$ belong to the separating family $\mathcal{T}^B$ of stable tubes of Γ_B, we know that $\mathscr{C}$ and $\mathscr{D}$ are orthogonal, generalized standard, and without external short paths. In particular, $\mathscr{C}$ and $\mathscr{D}$ do not lie on short cycles in Σ_B. Then, applying [23, Lemmas 3.1 and 3.3], we conclude that $\mathscr{C}$ and $\mathscr{D}$ consist of modules which do not lie on infinite short cycles in $\mathrm{mod}\, B$. Assume $\mathscr{C}$ is of rank $r \geq 2$. Take a module X lying on the mouth of $\mathscr{C}$ (X has one immediate predecessor and one immediate successor in $\mathscr{C}$). Then, by [23, Corollary 4.4], X is uniquely determined by $[X]$, which contradicts the fact that $[X] = [Y]$ for some module Y in $\mathscr{D}$ and $\mathscr{C} \neq \mathscr{D}$. Therefore, $\mathscr{C}$ is of rank one. Applying the same arguments, we conclude that $\mathscr{D}$ is also of rank one.

Summing up, the proofs of Theorems 1 and 2 are provided.

3. Examples

Let K be an algebraically closed field and Q be a finite quiver. For any arrow $\alpha \in Q$, by $s(\alpha)$ and $t(\alpha)$ we mean the source and the target of α, respectively. By KQ we denote the path algebra of Q. Recall that, if the quiver Q is acyclic, then KQ is a hereditary algebra [1]. For a finite-dimensional algebra H over K, we denote by $T(H)$ the trivial extension algebra of H by its duality H-H-bimodule $D(H) = \mathrm{Hom}_K(H, K)$. Recall that $T(H) = H \oplus D(H)$ as a K-vector space and the multiplication in $T(H)$ is given by $(a, f)(b, g) = (ab, ag + fb)$ for $a, b \in H$ and $f, g \in D(H)$. Then $T(H)$ is a symmetric algebra and H is the quotient algebra of $T(H)$ by the ideal $D(H)$.

For a natural number $n \geq 4$, Q_n will be the quiver of the following form:

$$\begin{tikzcd}
1 \arrow[dr, shift left] & & & & & & n+1 \arrow[dl, shift left] \\
& 3 \arrow[ul, shift left] \arrow[dl, shift left] \arrow[r, shift left] & 4 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & n-2 \arrow[l, shift left] \arrow[r, shift left] & n-1 \arrow[l, shift left] \arrow[ur, shift left] \arrow[dr, shift left] & \\
2 \arrow[ur, shift left] & & & & & & n \arrow[ul, shift left]
\end{tikzcd}$$

Each arrow in Q_n will be named either by α or by β in such a way that an arrow which starts in the vertex 3 and terminates in the vertex 1 is α, and $s(\alpha) = t(\beta)$, $t(\alpha) = s(\beta)$, for all arrows α and β. Let I_n be the admissible ideal in the path algebra KQ_n generated by all paths $\alpha\beta, \beta\alpha$ such that $s(\alpha\beta) \neq t(\alpha\beta), s(\beta\alpha) \neq t(\beta\alpha)$, and all commutativity relations $\omega_1 - \omega_2$, where ω_1, ω_2 are all paths of length 2 in Q_n such that their source and target coincide with the vertex i, for all $i \in \{3, \dots, n-1\}$. Then by Λ_n we denote the quotient algebra KQ_n/I_n.

We consider now the quiver Δ_n of Euclidean type $\widetilde{\mathbb{D}}_n$, for any $n \geq 4$, defined in the following way. If n is an odd number, then Δ_n is of the form:

$$\begin{tikzcd}
1 & & & & & & n+1 \arrow[dl] \\
& 3 \arrow[ul] \arrow[dl] \arrow[r] & 4 & \cdots \arrow[l] & n-2 \arrow[l] \arrow[r] & n-1 & \\
2 & & & & & & n \arrow[ul]
\end{tikzcd}$$

and similarly, for an even number n, the quiver Δ_n is of the form:

$$\begin{tikzcd}
1 & & & & & & n+1 \\
& 3 \arrow[ul] \arrow[dl] \arrow[r] & 4 & \cdots \arrow[l] \arrow[r] & n-2 & n-1 \arrow[l] \arrow[ur] \arrow[dr] & \\
2 & & & & & & n
\end{tikzcd}$$

(in particular, all maximal subquivers of type $\mathbb{A}_{n-1}$ of Q_n have an alternate orientation of arrows).

Let H_n be the path algebra $K\Delta_n$ and H_n^* the path algebra $K\Delta_n^*$, where Δ_n^* is the opposite quiver of Δ_n. Note that Δ_n is a subquiver of Q_n given by the arrows α and Δ_n^* is a subquiver of Q_n given by the arrows β. Moreover, observe that Λ_n is the trivial extension algebra $T(H_n)$ of H_n and the trivial extension algebra $T(H_n^*)$ of H_n^*. In particular, H_n and H_n^* are quotient algebras of Λ_n.

Assume now that $n > 4$ is an odd number. For each arrow α in Δ_n such that $s(\alpha) = i$ and $t(\alpha) \in \{i-1, i+1\}$, for some $i \in \{3, \dots, n\}$, we put α_l instead of α, where l is given by the formula:

$$l = \begin{cases} \frac{i-1}{2}, & \alpha : i \to i+1, \\ n - \frac{i+1}{2}, & \alpha : i \to i-1. \end{cases}$$

Observe that $l \in \{1, \dots, n-2\}$. We define the family of indecomposable representations $F_{\alpha_1}, \dots, F_{\alpha_{n-2}}$ of H_n over K:

- F_{α_l} for $l \notin \{\frac{n-1}{2}, n-2\}$:

$$\begin{tikzcd}
0 & & & & & & & 0 \arrow[dl] \\
& 0 \arrow[ul] \arrow[dl] \arrow[r] & \cdots \arrow[r, dash] & K \arrow[r, dash, "1"] & K \arrow[r, dash] & \cdots \arrow[r] & 0 & \\
0 & & & & & & & 0 \arrow[ul]
\end{tikzcd}$$

  where K stands in for the vertices $s(\alpha_l), t(\alpha_l)$, zero space elsewhere (here by $\longrightarrow$ with no arrowhead we mean $\longrightarrow$ or $\longleftarrow$);

- $F_{\alpha_{\frac{n-1}{2}}}$:

$$\begin{tikzcd}
0 & & & & & & K \arrow[dl, "1"'] \\
& 0 \arrow[ul] \arrow[dl] \arrow[r] & 0 & \cdots \arrow[l] & 0 \arrow[l] \arrow[r] & K & \\
0 & & & & & & K \arrow[ul, "1"]
\end{tikzcd}$$

where K stands in for the vertices $n-1, n, n+1$, zero space elsewhere;

- $F_{\alpha_{n-2}}$:

$$\begin{tikzcd}
K & & & & & & 0 \arrow[dl] \\
& K \arrow[ul, "1"'] \arrow[dl, "1"] \arrow[r] & 0 & \cdots \arrow[l] & 0 \arrow[l] \arrow[r] & 0 & \\
K & & & & & & 0 \arrow[ul]
\end{tikzcd}$$

where K stands in for the vertices $1, 2, 3$, zero space elsewhere.

Let $E_l = F_{\alpha_l}$ for $l \in \{1, \ldots, n-2\}$. Obviously $E_1, \ldots, E_{n-2}$ are pairwise orthogonal bricks. Direct calculation shows that $\tau_{H_n} E_{l+1} = E_l$ if $l \in \{1, \ldots, n-3\}$ and $\tau_{H_n} E_1 = E_{n-2}$. Moreover, $\mathrm{Ext}^2_{H_n}(E_r, E_p) = 0$ for any $r, p \in \{1, \ldots, n-2\}$, because H_n is a hereditary algebra. It allows us to state that $E_1, \ldots, E_{n-2}$ form the mouth of a standard stable tube $\mathcal{T}$ of rank $n-2$ in Γ_{H_n} (see [17], [20]). Since $\mathrm{pd}_{H_n} X \leq 1$ for any H_n-module X in $\mathcal{T}$, it follows from [26, Proposition 1.1] that $\mathcal{T}$ is also a component of the Auslander-Reiten quiver Γ_{Λ_n}.

Analogously, let $E_1^*, E_2^*, \ldots, E_{n-2}^*$ be the indecomposable H_n^*-modules, where the indices l are given in such a way that, for any $l \in \{1, \ldots, n-2\}$, E_l and E_l^* have the same composition factors in $\mathrm{mod}\,\Lambda_n$ including the multiplicities. It is easy to see that these modules form the mouth of a stable tube $\mathcal{T}^*$ of rank $n-2$ in $\Gamma_{H_n^*}$ such that $\tau_{H_n^*} E_l^* = E_{l+1}^*$ for $l \in \{1, \ldots, n-3\}$ and $\tau_{H_n^*} E_{n-2}^* = E_1^*$. Using once more [26, Proposition 1.1] we get that $\mathcal{T}^*$ is also a component of the Auslander-Reiten quiver Γ_{Λ_n}. Note that $\mathrm{top}(E_l^*) = \mathrm{soc}(E_l)$ and $\mathrm{top}(E_l) = \mathrm{soc}(E_l^*)$ in $\mathrm{mod}\,\Lambda_n$, for any $l \in \{1, \ldots, n-2\}$. Therefore, $\mathcal{T}$ has an external short path $E_l \to E_l^* \to E_l$ in $\mathrm{mod}\,\Lambda_n$, which implies existence of a short cycle $\mathcal{T} \to \mathcal{T}^* \to \mathcal{T}$ in Σ_{Λ_n}. Observe also that $\mathcal{T}$ and $\mathcal{T}^*$ have the same composition factors since $[E_l] = [E_l^*]$ for all $l \in \{1, \ldots, n-2\}$. Moreover, $\mathcal{T}$ and $\mathcal{T}^*$ are generalized standard stable tubes in Γ_{Λ_n} since they are generalized standard in Γ_{H_n} and $\Gamma_{H_n^*}$, respectively (see for example [20, Chapter X]).

Assume $n \geq 4$ is an even number. For each arrow α in H_n such that $s(\alpha) = i$ and $t(\alpha) \in \{i-1, i+1\}$, for some $i \in \{3, \ldots, n-1\}$, we define the index l in the previous way. Similarly, we define the indecomposable representations $F_{\alpha_1}, \ldots, F_{\alpha_{n-2}}$ of H_n over K:

- F_{α_l} for $l \notin \{\frac{n-2}{2}, n-2\}$:

$$\begin{tikzcd}
0 & & & & & & & 0 \\
& 0 \arrow[ul] \arrow[dl] \arrow[r] & \cdots \arrow[r, dash] & K \arrow[r, dash, "1"] & K \arrow[r, dash] & \cdots & 0 \arrow[l] \arrow[ur] \arrow[dr] & \\
0 & & & & & & & 0
\end{tikzcd}$$

where K stands in for the vertices $s(\alpha_l), t(\alpha_l)$, zero space elsewhere (here by $—$ we mean $\longrightarrow$ or $\longleftarrow$);

- $F_{\alpha_{\frac{n-2}{2}}}$:

$$\begin{tikzcd}
0 & & & & & & K \\
& 0 \arrow[ul] \arrow[dl] \arrow[r] & 0 & \cdots \arrow[l] \arrow[r] & 0 & K \arrow[l] \arrow[ur, "1"] \arrow[dr, "1"'] & \\
0 & & & & & & K
\end{tikzcd}$$

where K stands in for the vertices $n-1, n, n+1$, zero space elsewhere;

- $F_{\alpha_{n-2}}$:

$$\begin{tikzcd}
K & & & & & & 0 \\
& K \arrow[ul, "1"'] \arrow[dl, "1"] \arrow[r] & 0 & \cdots \arrow[l] \arrow[r] & 0 & 0 \arrow[l] \arrow[ur] \arrow[dr] & \\
K & & & & & & 0
\end{tikzcd}$$

where K stands in for the vertices $1, 2, 3$, zero space elsewhere.

As before the modules $E_l = F_{\alpha_l}$, $l \in \{1, \dots, n-2\}$, form the mouth of a stable tube $\mathcal{T}$ of rank $n-2$ in Γ_{H_n} in such a way that $\tau_{H_n} E_{l+1} = E_l$ for $l \in \{1, \dots, n-3\}$ and $\tau_{H_n} E_1 = E_{n-2}$. Similarly, let $\mathcal{T}^*$ be the stable tube of rank $n-2$ in $\Gamma_{H_n^*}$ whose mouth consists of the modules $E_1^*, E_2^*, \dots, E_{n-2}^*$, where the indices l are given in such a way that, for any $l \in \{1, \dots, n-2\}$, E_l and E_l^* have the same composition factors and $\operatorname{top}(E_l^*) = \operatorname{soc}(E_l)$, $\operatorname{top}(E_l) = \operatorname{soc}(E_l^*)$ in $\operatorname{mod} \Lambda_n$. Therefore, there is a short cycle $\mathcal{T} \to \mathcal{T}^* \to \mathcal{T}$ in Σ_{Λ_n}. Moreover, $\mathcal{T}$ and $\mathcal{T}^*$ are generalized standard components in Γ_{Λ_n}.

Summing up, we have proved that for an arbitrary $m \geq 2$, the Auslander-Reiten quiver $\Gamma_{\Lambda_{m+2}}$ of Λ_{m+2} contains a generalized standard stable tube of rank m which is not uniquely determined by its composition factors.

We end this section with comments concerning acyclic generalized standard Auslander-Reiten components. It has been proved in [22, Corollaries 2.4 and 3.3] that every acyclic generalized standard component $\mathscr{C}$ of the Auslander-Reiten quiver Γ_A of an Artin algebra A is of the form $\mathbb{Z}\Delta$ for a finite acyclic connected valued quiver Δ with at least three vertices, $B = A/\operatorname{ann}_A(\mathscr{C})$ is a tilted algebra of the form $\operatorname{End}_H(T)$, for some wild hereditary Artin algebra H and a regular tilting H-module, and $\mathscr{C}$ is the connecting component $\mathscr{C}_T$ of Γ_B determined by T. Moreover, C. M. Ringel proved in [18] that, for any connected wild hereditary Artin algebra H whose ordinary valued quiver has at least three vertices, there exists a multiplicity-free regular tilting module T in $\operatorname{mod} H$, and consequently the connecting component $\mathscr{C}_T$ of the Auslander-Reiten quiver Γ_B of the associated tilted algebra $B = \operatorname{End}_H(T)$ is an acyclic generalized standard faithful regular component of Γ_B. We refer also to [10] for constructions of tilted algebras having regular connecting components with arbitrary large composition factors.

Let K be an algebraically closed field, Q an arbitrary connected acyclic wild quiver with at least three vertices, and $H = KQ$. Then it follows from [10, Corollary 4] that there are infinitely many pairwise non-isomorphic tilted algebras $B = \operatorname{End}_H(T)$, for multiplicity-free regular tilting modules T in $\operatorname{mod} H$, such that the connecting component $\mathscr{C}_T$ determined by T is regular and without simple modules. Take such a tilted algebra $B = \mathrm{End}_H(T)$ and consider the trivial extension algebra $\Lambda = T(B)$ of B by the B-B-bimodule $D(B) = \mathrm{Hom}_K(B, K)$. Then it follows from [5, Section 5] that the Auslander-Reiten quiver Γ_Λ of Λ consists of two acyclic generalized standard regular sincere components $\mathscr{C} = \mathscr{C}_T$ and $\mathscr{D}$, having sections of type $\Delta = Q^{\mathrm{op}}$, and infinitely many components whose stable parts are of the form $\mathbb{Z}\mathbb{A}_\infty$. However, it is not clear if $\mathscr{C}$ and $\mathscr{D}$ may have the same composition factors. It would be interesting to know if such a situation may occur.

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