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PICARD'S THEOREM AND RICKMAN'S THEOREM BY WAY OF HARNACK'S INEQUALITY

JOHN L. LEWIS

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ABSTRACT. In this note we give a very elementary proof of Picard's Theorem and Rickman's Theorem which uses only Harnack's inequality.

1. INTRODUCTION

The classical theorem of Picard can be stated as:
A nonconstant entire analytic function in the complex plane omits at most one complex value.

Many proofs of this theorem have been given using (a) the modular function, (b) Schottky and Bloch theorems, (c) a generalization of Schwarz's lemma, (d) Nevannlinna's second fundamental theorem, and (e) probability (see [5] for (a), (b); see [1] for (c); see [6] for(d); and see [2] for (e)). Rickman [10] has obtained an analogue of Picard's theorem for entire quasiregular mappings. He proved that:
A nonconstant entire $K > 1$ quasiregular function in $\mathbb{R}^n$ omits at most $m = m(n, K)$ values.

In this note we show that both theorems are rather easy consequences of a Harnack-type inequality which can be stated as: If $x = (x_1, x_2, \dots, x_n)$ denotes a point of Euclidean n space $\mathbb{R}^n$, $B(x, 2r) = \{y \in \mathbb{R}^n : |y - x| < 2r\}$ and h is a nonnegative real-valued function defined on $B(x, 2r)$, then

$$(1.1)\quad M(r, h, x) = \sup\{h(y) : y \in B(x, r)\} \le \theta \inf\{h(y) : y \in B(x, r)\}$$

for some $\theta \ge 1$. A continuous function u defined on $\mathbb{R}^n$ will be called a *Harnack function* on $\mathbb{R}^n$ with constant θ provided that (1.1) holds for each $x \in \mathbb{R}^n$, $r > 0$, whenever h is nonnegative on $B(x, 2r)$ and h has the form $h = \pm u + a$ for some $a \in \mathbb{R}$. The key ingredient in our proof is

Lemma 1. *Let u be a Harnack function with constant θ, $u(x_0) = 0$, and $R > 0$. Then there exist r, $0 < r < R$, $x_1 \in B(x_0, 2R)$, and $c_1 = c_1(\theta) \ge 2$*

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such that $u(x_1) = 0$ *and*

$$(c_1)^{-1} M(R, u, x_0) \leq M(10r, u, x_1) \leq c_1 M(r, u, x_1).$$

We note that the above lemma in much less generality can be found in Rickman's paper [10] and also in Eremenko and Lewis's [3]. Both Rickman and Eremenko and Lewis prove Lemma 1 for the class of u that they consider by first showing that if $u(y) = 0$, then the maximum of u on a ball with center at y can be estimated above and below by the μ measure of a larger and smaller ball centered at y, respectively. In Rickman's case, μ is a certain average of the counting function for a given quasiregular entire function, while, in Eremenko and Lewis's case, μ is a certain positive Riesz measure associated with $u^+ = \max\{u, 0\}$. Second, these authors prove a lemma similar to Lemma 1 with $M(\cdot, u, \cdot)$ replaced by μ. Both proofs of Lemma 1 are somewhat involved and use nonlinear pde theory (Rickman also uses the theory of extremal length).

Proof. We obtain Lemma 1 directly by an embarassingly simple argument. Let $\delta(x) = 2R - |x - x_0|$ be the distance from $x \in B(x_0, 2R)$ to $\mathbb{R}^n \setminus B(x_0, 2R)$. Put $E = \{x : u(x) = 0\} \cap B(x_0, 2R)$, and let F be the closure of $\bigcup_{x \in E} B(x, \frac{\delta(x)}{100})$. Set

$$\gamma = \sup\{M(10^{-2}\delta(x), u, x) : x \in E\},$$

and choose x_1 in E such that if $r = \frac{\delta(x_1)}{100}$, then

$$\gamma \leq 2M(r, u, x_1). \tag{1.2}$$

We show that Lemma 1 is valid for x_1, r as above. Indeed since $|\delta(x) - \delta(y)| \leq |x - y|$ for $x, y, \in B(x_0, 2R)$, it follows easily that for $y \in B(x_1, 20r)$

$$\delta(x_1) \leq 2\delta(y) \leq 4\delta(x_1). \tag{1.3}$$

Let $\bar{K}$ denote the closure of the set K and choose $x_2 \in \bar{B}(x_1, 10r)$ with

$$M(10r, u, x_1) \leq 2u(x_2). \tag{1.4}$$

We now consider two cases. If $x_2 \in F$, then from (1.2), (1.4), and continuity of u we see that

$$M(10r, u, x_1) \leq 2u(x_2) \leq 2\gamma \leq 4M(r, u, x_1). \tag{1.5}$$

If $x_2 \notin F$, we use interval notation to denote the line segment connecting two points. Since F is closed, there exists $z \in (x_1, x_2) \cap F$ with $[x_2, z) \cap F = \varnothing$. We claim that each $w \in [x_2, z)$ contains a ball of radius $\frac{r}{4}$ centered at w on which $u \geq 0$. Otherwise, by continuity of u and (1.3) we would have $u(y) = 0$ for some y with

$$|y - w| \leq \frac{r}{4} = \frac{\delta(x_1)}{400} < \frac{\delta(y)}{100}.$$

Thus $w \in F$, which contradicts our choice of z. Hence our claim is true. Since $[x_2, z]$ can be covered by at most 80 balls of radius $\frac{r}{8}$ with centers in $[x_2, z)$, we can use Harnack's inequality (1.1) recursively and (1.4) to conclude that

$$M(10r, u, x_1) \le 2u(x_2) \le 2\theta^{80} u(z) \le 4\theta^{80} M(r, u, x_1).$$

Here the right-hand inequality follows from the fact that $z \in F$ and the same argument as in (1.5). Thus if $c_1 = 4\theta^{80}$, then in either case we have

$$M(10r, u, x_1) \le c_1 M(r, u, x_1),$$

which is the right-hand inequality in Lemma 1. The proof of the left-hand inequality in Lemma 1 is similar. That is, choose $x_3 \in \bar{B}(x_0, R)$ such that $u(x_3) \ge 2M(R, u, x_0)$. Either $x_3 \in F$ or $x_3 \notin F$, and in either case arguing as above we get the left-hand inequality in Lemma 1. The proof of Lemma 1 is now complete. □

Next we use Lemma 1 and elementary properties of harmonic functions to give a short proof of Picard's Theorem for analytic functions. In the proof we shall often justify our reasoning with such statements as "by well-known properties of harmonic functions in $\mathbb{R}^2$". We do this so that the reader well versed in harmonic functions can rapidly see how Picard's theorem follows from Lemma 1. Later we shall see that the only properties of harmonic functions which we use, except for real analyticity, are easily obtainable from the fact that harmonic functions are Harnack functions.

Proof. The proof of Picard's Theorem is by contradiction. We identify the complex plane with $\mathbb{R}^2$ in the usual way. Suppose F is an entire nonconstant analytic function which omits distinct complex numbers a_1, a_2. Then $f = \frac{F-a_1}{a_2-a_1}$ is a nonconstant entire function in the complex plane ($\mathbb{R}^2$) which omits 0 and 1. Put $u_1 = \log|f| - 2$, $u_2 = \log|f-1| - 2$, and observe that u_1, u_2 are harmonic in $\mathbb{R}^2$. From Harnack's theorem for positive harmonic functions we deduce that u_1, u_2 are Harnack functions for some constant θ. Since nonconstant harmonic functions on $\mathbb{R}^2$ are unbounded above and below (as is well known), we can choose $x_0 \in \mathbb{R}^2$ with $u_1(x_0) = 0$ and apply Lemma 1 with $R = 2^j$, $j = 1, 2, \ldots,$ to get sequences $\{z_j\}, \{r_j\}$ with

$$(1.6)\qquad \begin{aligned} &(\alpha) \quad \lim_{j\to\infty} M(r_j, u_1, z_j) = \infty, \\ &(\beta) \quad M(10r_j, u_1, z_j) \le c_1 M(r_j, u_1, z_j), \\ &(\gamma) \quad u_1(z_j) = 0 \end{aligned}$$

for $j = 1, 2, \ldots$. Define $v_{1,j}, v_{2,j}$, $j = 1, 2, \ldots$, on $B(0, 1)$ by $v_{i,j}(z) = u_i(z_j + 10r_j z)/M(10r_j, u_1, z_j)$ for $i = 1, 2$ when $z \in B(0, 1)$. Using (1.6) (α)–(γ) and properties of harmonic functions, it is easily seen that a subsequence of $\{v_{i,j}\}$ converges for $i = 1, 2$ uniformly on compact subsets of $B(0, 1)$ to harmonic functions v_1, v_2, respectively, in $B(0, 1)$. Moreover, from (1.6) (α)–(γ) it can be deduced that

$$(1.7)\qquad \begin{aligned} &(*) \quad v_i(0) = 0 \text{ for } i = 1, 2, \\ &(**) \quad v_1 = v_2 \text{ on } \textstyle\bigcup_{i=1}^2 \{x : v_i(x) > 0\} \ne \varnothing, \\ &(***) \quad \{x : v_1(x) < 0\} \cap \{x : v_2(x) < 0\} = \varnothing. \end{aligned}$$

Since harmonic functions are real analytic, it follows from (1.7)$(**)$ that $v_1 \equiv v_2$. From (1.7)$(***)$ it then follows that $v_1 \ge 0$. Using Harnack's inequality, (1.1), and (1.7)($*$), we then conclude that $v_1 \equiv 0$, which contradicts (1.7)($**$). From this contradiction we conclude Picard's theorem. □

We note that Lemma 1 was used only to insure that $v_1 \not\equiv 0$. In fact, from (1.6)(β) and uniform convergence it is easily seen that $M(\frac{1}{10}, v_1, 0) \geq (c_1)^{-1}$. We also note that our proof used the limit function technique of Eremenko and Sodin (see [4, 14]).

2. Proof of Rickman's Theorem

For completeness we give a rapid review of quasiregular mappings (see [9, 12]). Recall that a function $f = (f_1, f_2, \dots, f_n)$ from a domain $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$ is said to be $K > 1$ quasiregular provided each of its coordinate functions has a distributional partial derivative in Ω which is locally nth power integrable and

$$(2.1) \qquad |Df(x)|^n = \sup_{|h|=1} |Df(x)h|^n \leq KJ_f(x)$$

for almost every $x \in \Omega$ with respect to Lebesgue n measure. In (2.1) $Df(x)$ is the Jacobian matrix of f, while J_f is the determinant of Df (the Jacobian of f). Moreover, if $f \neq d$ in Ω, then $u = \log|f - d|$ is locally a weak solution to

$$(2.2) \qquad \nabla \cdot [\langle A(x)\nabla u(x), \nabla u(x)\rangle^{(n-2)/2} A(x)\nabla u(x)] = 0$$

in Ω, where $A(x) = J_f(x)^{2/n}[D^t f(x) Df(x)]^{-1}$, when $Df(x)^{-1}$ exists, and $A(x)$ = identity matrix, otherwise. Here, $D^t f$ denotes the transpose matrix of Df. That is, if

$$A(x, \eta) = \langle A(x)\eta, \eta\rangle^{(n-2)/2} A(x)\eta,$$

when $(x, \eta) \in \Omega \times \mathbb{R}^n$, then

$$\int_\Omega \langle A(x, \nabla u), \nabla\phi \rangle \, dx = 0, \quad \text{whenever } \phi \in C_0^\infty(\Omega),$$

where $\langle \cdot, \cdot \rangle$ denotes inner product and dx denotes Lebesgue n measure. We note for some $c_2 = c_2(n, K) \geq 2$ that

(a) $(c_2)^{-1}|\eta|^n \leq \langle A(x, \eta), \eta\rangle$ for almost every $x \in \Omega$, whenever $\eta \in \mathbb{R}^n$,
(b) $|A(x, \eta)| \leq c_2|\eta|^{n-1}$ for almost every $x \in \Omega$, whenever $\eta \in \mathbb{R}^n$.

Serrin [13] has considered weak solutions to elliptic equations of the form (2.2) under more general structure conditions than (a) and (b). He showed that positive solutions satisfy a Harnack inequality, where the constant θ depends only on n and the structure constant c_2 in (a), (b).

Proof. Now suppose f is a nonconstant entire $K > 1$ quasiregular mapping which omits distinct values $a_1, \dots, a_m$. Let $b = \sum_{i=1}^m |a_i| + 1$, and put $u_i = \log|f - a_i| - b$, $1 \leq i \leq m$. Then each function of the form $\pm u_i + a$, where $a \in \mathbb{R}$ and $1 \leq i \leq m$, is a weak solution in $\mathbb{R}^n$ to a nonlinear divergence form elliptic equation satisfying structure conditions (a) and (b). From Serrin's theorem we deduce that each u_i, $1 \leq i \leq m$, is a Harnack function with constant θ depending only on n and K. To prove Rickman's Theorem we shall again use Lemma 1 and the limit function technique of Eremenko and

Sodin. Here, though the limit functions need not be real analytic, we will need an additional counting argument to conclude that there can be at most $m = m(n, K)$ distinct limiting functions. We shall need some simple properties of Harnack functions, similar to those used in the proof of Picard's Theorem.

Suppose w is a Harnack function with constant θ. Then there exist $\alpha = \alpha(\theta)$, $0 < \alpha < 1$, and $c = c(\theta) \geq 1$ such that

(2.3)
$$\operatorname*{osc}_{B(x,t_1)} w = \sup_{y,z\in B(x,t_1)} |w(y) - w(z)| = M(t_1, w, x) + M(t_1, -w, x)$$

$$\leq c(t_1/t_2)^\alpha \operatorname*{osc}_{B(x,t_2)} w = c(t_1/t_2)^\alpha[M(t_2, w, x) + M(t_2, -w, x)],$$

when $0 < t_1 < t_2$. To obtain (2.3) from Harnack's inequality (1.1), suppose $2r \leq t_2$ and let $h = M(2r, w, x) - w$, $h = M(2r, -w, x) + w$, respectively. Since $h \geq 0$ in $B(x, 2r)$ and w is a Harnack function with constant θ, we can apply Harnack's inequality to each h. Adding the resulting inequalities, we find for $\delta = \frac{\theta-1}{\theta+1}$ that

$$M(r, w, x) + M(r, -w, x) \leq \delta[M(2r, w, x) + M(2r, -w, x)].$$

Iterating this inequality starting with $2r = t_2$, we deduce that (2.3) is true. Similarly, if $h = M(2t, w, x) - w$, then from Harnack's inequality with $r = t$ we have

$$M(2t, w, x) + M(t, -w, x) \leq \theta[M(2t, w, x) - w(x)].$$

Thus

$$M(t, -w, x) \leq (\theta - 1)M(2t, w, x) - \theta w(x). \tag{2.4}$$

Equation (2.4) is also valid with w replaced by $-w$, since $-w$ is also a Harnack function. Equations (2.3) and (2.4) imply that if w is nonconstant, then

$$\lim_{t\to\infty} M(t, \pm w, x) = \infty. \tag{2.5}$$

Indeed, from (2.3) and (2.4) we have

$$\operatorname*{osc}_{B(x,t_1)} w \leq c(t_1/t_2)^\alpha \operatorname*{osc}_{B(x,t_2)} w \leq \theta\, c\, (t_1/t_2)^\alpha[M(2t_2, w, x) - w(x)],$$

Letting $t_2 \to \infty$ and using the fact that w is nonconstant, we get (2.5) for w. A similar argument can be given for $-w$.

We now proceed as in the proof of Picard's Theorem. From (2.5) we see that u_1 is unbounded above and below in $\mathbb{R}^n$. Thus we can choose $x_0 \in \mathbb{R}^n$ with $u_1(x_0) = 0$ and apply Lemma 1 again with $R = 2^j$, $j = 0, 1, 2, \dots$, to obtain sequences $\{z_j\}$, $\{r_j\}$ for which (1.6) holds. Put

$$v_{i,j}(x) = \frac{u_i(z_j + 10r_j x)}{M(10r_j, u_1, z_j)}, \qquad x \in B(0, 1),$$

and observe that $v_{i,j}$ is a Harnack function with constant θ for $1 \leq i \leq m$, $j = 1, 2, \dots$. From (1.6)(α) and the definition of u_i it is easily seen for

$1 \le i \le m$ that

$$M(t_j, u_i, z_j) - M(t_j, u_1, z_j) \to 0 \quad \text{as } j \to \infty, \tag{2.6}$$

whenever $t_j \ge r_j$. If $t_j = r_j$ in (2.6), then from (1.6)(β) we see for N large enough that

$$(2c_1)^{-1} \le M(\tfrac{1}{10}, v_{i,j}, 0) \le 2 \tag{2.7}$$

for $j \ge N$ and $1 \le i \le m$. Using (2.6) with $t_j = 10r_j$, (2.7), and Harnack's inequality applied to $2 - v_{i,j}$ for $j \ge N$ and $1 \le i \le m$, we deduce that $\{v_{i,j}\}_{j \ge N}$ is bounded on compact subsets of $B(0, 1)$. This fact, (2.3), and Ascoli's theorem imply that there exists a subsequence of $\{v_{i,j}\}_{j \ge N}$ which converges uniformly to v_i, $1 \le i \le m$, on compact subsets of $B(0, 1)$. We claim that (compare with (1.7))

$$\begin{aligned} &(*) \quad v_i(0) = 0, \quad \text{for } 1 \le i \le m, \\ (2.8) \quad &(**) \quad v_1 = v_i, \ 1 \le i \le m, \ \text{on } \textstyle\bigcup_{j=1}^{m} \{x : v_j(x) > 0\} \neq \varnothing, \\ &(***) \quad \{x : v_l(x) < 0\} \cap \{x : v_j(x) < 0\} = \varnothing, \qquad 1 \le l \neq j \le m. \end{aligned}$$

To prove (2.8)($*$) observe from (1.6)(γ) that for $1 \le i \le m$ and $j = 1, 2, \ldots$,

$$\begin{aligned} 1 \le e^b - |a_1 - a_i| &= |f(z_j) - a_1| - |a_1 - a_i| \\ &\le |f(z_j) - a_i| \le |f(z_j) - a_1| + |a_1 - a_i| \le e^b + b. \end{aligned}$$

Taking logarithms we deduce first that $-b \le u_{i,j}(z_j) \le \log 2$ and thereupon that (2.8)($*$) is true. To prove (2.8)($**$) observe from uniform convergence, (1.6)(β), (2.6), and (2.7) that v_i is continuous with

$$(2c_1)^{-1} \le M(\tfrac{1}{10}, v_i, 0) \le M(1, v_i, 0) \le 1 \tag{2.9}$$

for $1 \le i \le m$. Hence the union in (2.8)($**$) is nonempty. Moreover, if $v_l(x) > 0$ for some $x \in B(0, 1)$, then from (1.6)(α) we see that $u_{l,j}(x) \to \infty$ for j in a certain subsequence, say $j \in \{j_k\}$. It follows from the definition of $\{u_{i,j}\}$ that $u_{l,j}(x) - u_{i,j}(x) \to 0$ for $1 \le i \le m$ as $j \to \infty$ in $\{j_k\}$. From this fact and (1.6)(α) we conclude that (2.8)($**$) is true. To prove (2.8)($***$) note that if $v_l(x) < 0$ for some $x \in B(0, 1)$, then $|f(z_j + 10r_j x) - a_l| \to 0$ as $j \to \infty$ through a certain sequence. Thus, $u_{i,j}(x)$, $i \neq l$, is bounded below as $j \to \infty$ in this sequence, which in view of (1.6)(α) implies (2.8)($***$). We are now at the point where we used real analyticity in Picard's Theorem. Since v_i, $1 \le i \le m$, may not be real analytic, we need another argument. We show in fact that

$$c_3 |\{x : v_i(x) < 0\}| \ge 1 \tag{2.10}$$

for some $c_3 = c_3(\theta) \ge 1$, where $|E|$ denotes the Lebesgue n measure of $E \subset B(0, 1)$. Using (2.8)($***$), we then conclude that $m \le c(\theta) = c(n, K)$, which is Rickman's Theorem. To prove (2.10) observe from (2.8)($*$), (2.4) with $w = -v_i$, $x = 0$, and (2.9) that

$$M(\tfrac{1}{5}, -v_i, 0) \ge \tau > 0 \tag{2.11}$$

for some $\tau = \tau(\theta)$, $0 < \tau < 1$, and $1 \le i \le m$. Choose $y_i \in \bar{B}(0, \frac{1}{5})$, $1 \le i \le m$, with $v_i(y_i) = -M(\frac{1}{5}, -v_i, 0)$. Put $w_i(x) = v_i(x) - v_i(y_i)$ when $1 \le i \le m$ and $x \in B(y_i, \frac{1}{2})$. We note that (2.3) holds for $w = w_i$, whenever $B(x, t_2) \subset B(0, 1)$, as follows from uniform convergence of a subsequence of $\{v_{i,j}\}$ to v_i, $1 \le i \le m$. Likewise (2.4) is also valid for w_i. Using (2.3), (2.4), and (2.9), we deduce for some $c_4 = c_4(\theta)$ that

$$M(t, w_i, y_i) \le c_4 t^\alpha, \qquad 0 < t < \tfrac{1}{4}. \tag{2.12}$$

From (2.11) and (2.12) we deduce the existence of $\epsilon_0 = \epsilon_0(\theta)$, $0 < \epsilon_0 \le \frac{1}{4}$, such that

$$M(\epsilon_0, w_i, y_i) \le -\tfrac{1}{2} v_i(y_i) = \tfrac{1}{2} M(\tfrac{1}{5}, -v_i, 0).$$

From this inequality and (2.11) we find that

$$v_i(x) \le \tfrac{1}{2} v_i(y_i) = -\tfrac{1}{2} M(\tfrac{1}{5}, -v_i, 0) \le -\tfrac{1}{2}\tau,$$

when $x \in B(y_i, \epsilon_0)$ and $1 \le i \le m$.

Thus (2.10) holds for c_3 sufficiently large. The proof of Rickman's Theorem is now complete. □

3. Remarks about Harnack functions

Suppose v_i, $1 \le i \le m$, are nonconstant Harnack functions which satisfy (2.8) and $m \ge 2$. Let $O_1 = \{x : v_1(x) > 0\}$, and put $O_i = \{x : v_{i-1}(x) < 0\}$ for $2 \le i \le m+1$. Using Harnack's inequality, it is easily seen that $\partial O_i = \partial O_j$, $1 \le i, j \le m+1$. Such an equality between open sets is topologically possible (e.g., the lakes of Wada), but it requires some work to construct examples of such sets. Thus a natural question to ask is whether $m \ge 2$ can occur in $\mathbb{R}^2$. If not, then Picard's Theorem could be proved using only simple properties of Harnack functions. In $\mathbb{R}^3$, Rickman [11] has constructed some examples which show that for a given positive integer $m \ge 2$ there exists for K sufficiently large a K quasi-regular entire function which omits m values in $\mathbb{R}^3$. Moreover, these examples can be used as in the proof of Rickman's Theorem to show that $m \ge 2$ can occur in (2.8). In this case, though, one can ask further questions, such as do there exist $m \ge 2$ Harnack functions satisfying (2.8) and also any reasonable pde (for example, those considered by Serrin)? It is not difficult to see, for example, that if the v's are C^1 solutions to an elliptic pde (either in divergence or nondivergence form) for which a Hopf boundary maximum principle holds, then necessarily $m = 1$ in (2.8). At any rate any relatively simple examples of (2.8) for which $m \ge 2$ would be interesting.

We note that Harnack functions could be defined on a metric space and a version of Lemma 1 could be proved in this space provided that each ball of say radius $400r$ in this space contains at most N disjoint balls of radius r. Thus it is natural to ask when does Rickman's theorem hold for quasiregular mappings $f : \mathcal{M} \to \mathcal{N}$, where $\mathcal{M}$, $\mathcal{N}$ are Riemannian n manifolds? This question has been considered by Holopainen and Rickman [7, 8]. They used the technique in Eremenko and Lewis to obtain an analogue of Rickman's Theorem when $\mathcal{M}$ is $\mathbb{R}^n$ or $n = 2k+1$ and M is the n-dimensional Heisenberg group H_k, while $\mathcal{N} = \mathbb{R}^n \setminus \{a_1, a_2, \dots, a_m\}$, endowed with any Riemannian metric. The case $\mathcal{M} = \mathbb{R}^n$ completely answered a question of Gromov. Lemma 1 and the limit function technique of Eremenko and Sodin can be used to considerably simplify the work of these authors, and conceivably they also will lead to an extension of Rickman's Theorem to more general classes of functions and spaces. Finally, we note that an analogue of Schottky's theorem can also be obtained using Lemma 1, as in Eremenko and Lewis's paper [3] or by the limit function technique above.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY, 40506 - 0027

E-mail address: john@ms.uky.edu