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UNIFORMLY PERSISTENT SYSTEMS

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ABSTRACT. Conditions are given under which weak persistence of a dynamical system with respect to the boundary of a given set implies uniform persistence.

1. Introduction. Frequently in mathematical modeling, systems of ordinary differential equations are used in the study of the dynamical behaviour of entities which for the sense of the given context must remain nonnegative for all times. Typically, then, the arena for such systems is the nonnegative cone in $\mathbf{R}^n$. The boundary of the cone is a "barrier" for the dynamical system in question; in many cases it is also invariant so that entities that are zero at any time remain so for all times. One has then a system of the form

$$(1.1)\qquad \begin{cases} x_i' = x_i f_i(x_1, x_2, \ldots, x_n); & i = 1, \ldots, n, \\ x_i(0) \geq 0; & ('= d/dt), \end{cases}$$

in the case of an autonomous system where each f_i is smooth enough to guarantee uniqueness of solutions to initial value problems, together with other properties appropriate for the physical, biological, etc., system being modeled.

One question that arises is that of determining conditions which prevent solutions of (1.1) which are initially strictly positive from approaching the boundary of the cone as time evolves. This is of paramount importance in the modeling of biological populations where such conditions rule out the possibility of one of the populations becoming arbitrarily close to zero in a deterministic model and therefore risking extinction in a more realistic interpretation of the model. Generally speaking the term *persistence* is given to systems in which strictly positive solutions do not approach the boundary of the nonnegative cone as $t \to \infty$. Various precise definitions of persistence have been given: a version of (*weak*) *persistence* [**5**, **6**, **7**] applied when it is required only that positive solutions do not asymptotically approach the boundary as $t \to \infty$; *persistence* [**3**, **4**] means that each strictly positive solution is eventually at some positive distance from the boundary; *uniform persistence*, also called cooperativeness or permanent coexistence [**10**, **12**], means that strictly positive solutions are eventually uniformly bounded away from the boundary. Weak persistence has a drawback in that it guarantees only that extinction is not certain. A priori, solutions may still approach arbitrarily close to the boundary

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for large values of t, and small stochastic perturbations applied to the model may result in the solution being driven to the boundary.

The notion of uniform persistence avoids this problem and may be expected to be a more robust concept, that is, more likely to be maintained under suitable small variations of the system of equations. This is always a desirable consideration from the point of view of applications. However, it is the case that, generally speaking, it is easier to find conditions that guarantee persistence (or weak persistence), rather than uniform persistence. It is the intention of this paper to bridge the gap between these concepts by showing that it is frequently the case that weak persistence implies uniform persistence. Although we have mentioned biological models as a motivation for these concepts, it is easy to conceive other modeling situations in which some form of persistence is a useful idea. Bearing in mind other possible applications, we shall place our definitions and analysis in a more general setting than that of the previous discussion, namely, for a continuous flow on a locally compact metric space with boundary.

2. Definitions, notation and preliminaries. Let $\mathcal{E}$ be a locally compact metric space with metric d. For any subset S of $\mathcal{E}$, we shall use $\mathring{S}, \partial S, \overline{S}$ to denote its interior, boundary and closure, respectively. We shall consider a continuous flow $\mathcal{F} = (E, \mathbf{R}, \pi)$ on E, where E is a closed subset of $\mathcal{E}$, $\mathbf{R}$ is the real numbers and $\pi(x,t)$ is a continuous map from $E \times \mathbf{R}$ to E such that $\pi(\pi(x,t),s) = \pi(x,t+s)$ for all $x \in E,\ s,t \in \mathbf{R}$.

We assume that ∂E is invariant for $\mathcal{F}$; note that this implies that $\mathring{E}$ is also invariant and we denote the restriction of $\mathcal{F}$ to ∂E by $\partial\mathcal{F}$. We refer to [**1**] for all of the basic definitions, ideas and results for such flows; here we present only the notation we require and definitions which may not be entirely standard.

Notation. For any $x \in E$, the orbit, positive semiorbit and negative semiorbit of $\mathcal{F}$ through x are denoted by $\gamma(x), \gamma^+(x), \gamma^-(x)$, respectively, and the omega and alpha limit sets of $\gamma(x)$ are denoted by $\Lambda^+(x),\ \Lambda^-(x)$, respectively.

DEFINITIONS. An *isolated invariant set* M for the flow $\mathcal{F}$ is a nonempty invariant set which is the maximal invariant set in some neighbourhood of itself. Note that if M is a compact, isolated invariant set, one may always choose a compact isolating neighbourhood.

The *stable set* $W^+(M)$ of an isolated invariant set M is defined to be

$$\{x \in E: \Lambda^+(x) \neq \emptyset,\ \Lambda^+(x) \subset M\}$$

and the *unstable set* $W^-(M)$ is similarly defined in terms of $\Lambda^-(x)$. In [**1**] these sets are called regions of attraction and repulsion respectively. (Note that we assume no special structure for $M, W^+(M)$ or $W^-(M)$, but when E is a smooth manifold and M is, for example, a critical point, periodic orbit or periodic surface with hyperbolic structure, well-known results show that $W^+(M)$ and $W^-(M)$ have (local) manifold structure [**9, 14, 16**].)

The flow $\mathcal{F}$ will be called *dissipative* if for each $x \in E,\ \Lambda^+(x) \neq \emptyset$, and the invariant set $\Omega(\mathcal{F}) = \bigcup_{x\in E} \Lambda^+(x)$ has compact closure [**8**].

We shall say that $\mathcal{F}$ is *weakly persistent* if for all $x \in \overset{\circ}{E}$, $\overline{\lim}_{t\to\infty} d(\pi(x,t), \partial E) > 0$; is *persistent* if for all $x \in \overset{\circ}{E}$, we have $\underline{\lim}_{t\to\infty} d(\pi(x,t), \partial E) > 0$; is *uniformly persistent* if there exists $\varepsilon_0 > 0$ such that for all $x \in \overset{\circ}{E}$, $\underline{\lim}_{t\to\infty} d(\pi(x,t), \partial E) \geq \varepsilon_0$.

If $\mathcal{F}$ is dissipative, then $\partial\mathcal{F}$ is also dissipative and $\overline{\Omega(\partial\mathcal{F})}$ is a compact, isolated invariant set for $\partial\mathcal{F}$, which may or may not be isolated for $\mathcal{F}$. We shall call the flow $\partial\mathcal{F}$ *isolated* if there does exist a finite covering $\mathcal{M}$ of $\Omega(\partial\mathcal{F})$ by pairwise-disjoint compact isolated invariant sets $M_1, M_2, \dots, M_k$ for $\partial\mathcal{F}$ such that each M_i is also isolated for $\mathcal{F}$. $\mathcal{M}$ will then be called an *isolated* covering. If M, N are isolated invariant sets for $\mathcal{F}$, not necessarily distinct, we shall say that M is *chained* to N, and write this as $M \to N$, if there exist $x \notin M \cup N$ such that $x \in W^-(M) \cap W^+(N)$.

A *chain* of isolated invariant sets is a finite sequence $M_1, M_2, \dots, M_k$ with $M_1 \to M_2 \to \cdots \to M_k$ ($M_1 \to M_1$, if $k = 1$). The chain is called a *cycle* if $M_k = M_1$. Finally, $\partial\mathcal{F}$ will be called *acyclic* if for some isolated covering $\mathcal{M}$ of $\Omega(\partial\mathcal{F})$, $\mathcal{M} = \bigcup_{i=1}^k M_i$, no subset of $\{M_i\}$ forms a cycle.

REMARK. Our definition of cycle is very close to that of a k-cycle, for example in [**14**], but we do not necessarily require the invariant sets to be "transitive."

3. The main result.

THEOREM. *Let $\mathcal{F}$ be a continuous flow on a locally compact metric space E with metric d and boundary ∂E. Let $\partial\mathcal{F}$ be the restriction of $\mathcal{F}$ to ∂E (assumed invariant under $\mathcal{F}$). Assume that*

(H-1) *$\mathcal{F}$ is dissipative;*

(H-2) *$\mathcal{F}$ is weakly persistent;*

(H-3) *$\partial\mathcal{F}$ is isolated;*

(H-4) *$\partial\mathcal{F}$ is acyclic.*

Then $\mathcal{F}$ is uniformly persistent.

PROOF. Suppose not. Then there exist omega limit sets $\omega_n \subset \overset{\circ}{E}$ such that $d(\omega_n, \partial E) \to 0$ as $n \to \infty$. Without loss of generality we may assume that each ω_n is the closure of a full orbit, i.e. $\omega_n = \overline{\gamma_n}$. Note that the sequence ω_n is a uniformly bounded sequence of nonempty, compact subsets of $\overline{\Omega(\mathcal{F})}$ which is compact by (H-1). Let $\mathcal{K}$ be the metric space of nonempty, compact subsets of $\overline{\Omega(\mathcal{F})}$ with the Hausdorff metric ρ. Then $\mathcal{K}$ is compact [**17**]. Hence we may choose a convergent subsequence, which we relabel ω_n, such that $\rho(\omega_n, \omega) \to 0$ as $n \to \infty$, where $\omega \in \mathcal{K}$. Clearly ω is invariant and $d(\omega, \partial E) = 0$; this implies that $b = \omega \cap \partial E$ is a nonempty, compact invariant subset of ∂E. Thus $b' = b \cap \Omega(\partial\mathcal{F})$ is a nonempty, compact invariant set. By (H-3), we may find an acyclic set $M_1, M_2, \dots, M_k$ of compact invariant sets for $\partial\mathcal{F}$ such that each M_i is isolated for $\mathcal{F}$ and $\Omega(\partial\mathcal{F}) \subset \bigcup_{i=1}^k M_i$. Choose compact isolating neighbourhoods U_i, V_i of M_i in E such that $U_i \subset \overset{\circ}{V}_i$, and such that the V_i are pairwise disjoint, $i = 1, \dots, k$.

Let $b_1 = b' \cap M_{i_1} \neq \varnothing$. Then for n sufficiently large, $\omega_n \cap V_{i_1} \neq \varnothing$.

If we had $\omega_n \subset U_{i_1}$, then since $\omega_n \cap \partial E \neq \varnothing$, $\omega_n \cup M_{i_1}$ would be an invariant set contained in U_{i_1}, properly containing M_{i_1}, which would violate the isolating property. Thus for n sufficiently large, we can find points $x_{1n}, z_{1n} \in \gamma_n$ which are "entry" and "exit" points for the orbit γ_n and times s_{1n}, t_{1n} with $0 < s_{1n} < t_{1n}$, such that $\pi(x_{1n}, [0, t_{1n}]) \subset U_{i_1}$, $x_{1n} \in \partial U_{i_1}$, $z_{1n} = \pi(x_{1n}, t_{1n}) \in \partial U_{i_1}$, and such that $y_{1n} = \pi(x_{1n}, s_{1n})$ satisfies $d(y_{1n}, b_1) \to 0$ as $n \to \infty$.

Claim 1. $d(x_{1n}, \partial E) \to 0$ as $n \to \infty$. If not, there exists a subsequence which we again label x_{1n}, with $x_{1n} \to \tilde{\xi}_1 \in \partial U_{i_1} \cap \overset{\circ}{E}$. Without loss of generality we may also assume that $y_{1n} \to \tilde{\eta}_1 \in b_1 \subset M_{i_1}$. Consider the semiorbit $\gamma^+(\tilde{\xi}_1)$. If it is contained in V_{i_1}, then $\Lambda^+(\tilde{\xi}_1) \subset V_{i_1}$ and so either $\Lambda^+(\tilde{\xi}_1) \subset M_{i_1}$, contradicting (H-2), or $\Lambda^+(\tilde{\xi}_1) \cup M_{i_1}$ is an invariant set that violates the isolating property of V_{i_1}. Thus we may suppose that $\gamma^+(\tilde{\xi}_1)$ "exits" V_{i_1} for the first time at $t = \tau_1$. Then $\tilde{\zeta}_1 = \pi(\tilde{\xi}_1, \tau_1) \in \partial V_{i_1}$. Continuity of π shows that $\pi(x_{1n}, \tau_1) \to \tilde{\zeta}_1$ as $n \to \infty$. Since $\tilde{\zeta}_1$ is exterior to U_{i_1}, we must have $0 < s_{1n} < t_{1n} < \tau_1$. We may now assume that $s_{1n} \to \sigma$, say, where $0 < \sigma < \tau_1$ and $\pi(\tilde{\xi}_1, \sigma) = \tilde{\eta}_1 \in \partial E$. Since $\tilde{\xi}_1 \in \overset{\circ}{E}$, this violates the invariance of $\overset{\circ}{E}$, which proves the claim.

Thus we may suppose that $x_{1n} \to \xi_1$ as $n \to \infty$, where $\xi_1 \in \partial U_{i_1} \cap \partial E$. Consider the orbit $\gamma(\xi_1)$.

Claim 2. $\Lambda^+(\xi_1) \subset M_{i_1}$. The argument used to verify Claim 1 shows that, if $\gamma^+(\xi_1)$ ever "exits" V_{i_1}, then it contains a point of b_1, which is impossible since b_1 is an invariant subset of $\overset{\circ}{V}_{i_1}$ and $\xi_1 \in \partial V_{i_1}$ which is disjoint from b_1. Thus $\gamma^+(\xi_1) \subset V_{i_1}$. Since V_{i_1} is an isolating neighborhood for M_{i_1}, the only remaining possibility is that $\Lambda^+(\xi_1) \subset M_{i_1}$ as claimed.

$\gamma^-(\xi_1)$ is bounded, otherwise the semiorbits γ_n^- would fail to be uniformly bounded. Hence $\Lambda^-(\xi_1)$ is a nonempty, compact invariant subset of ∂E. Thus it has nonempty intersection with $\Omega(\partial \mathcal{F})$. Suppose then that $\Lambda^-(\xi_1) \cap M_{i_2} \neq \emptyset$. Since the points x_{1_n} on the orbits γ_n satisfy $x_{1n} \to \xi_1$ as $n \to \infty$, and since the $\omega_n = \overline{\gamma}_n \to \omega$ (in the Hausdorff metric) as $n \to \infty$, it follows, using the continuity of π, that $\Lambda^-(\xi_1) \subset \omega$ and so $b_2 = b' \cap M_{i_2} \neq \emptyset$. Exactly parallel arguments as used to verify Claim 2 but with time reversal and the use of negative semiorbits rather than positive semiorbits shows that in fact we have $\Lambda^-(\xi_1) \subset M_{i_2}$. Thus $\xi_1 \in W^-(M_{i_2}) \cap W^+(M_{i_1})$. Since $\xi_1 \in \partial V_{i_1}$, we have $\xi_1 \notin M_{i_2} \cup M_{i_1}$, and so $M_{i_2} \to M_{i_1}$. Since $b' \cap M_{i_2} \neq \emptyset$, we may repeat the preceding arguments to obtain M_{i_3} with $M_{i_3} \to M_{i_2}$. Since there are only finitely many sets M_i, continued repetition of the above argument must eventually lead to a cycle $M_{i_k} \to \cdots \to M_{i_j}$, $M_{i_j} = M_{i_k}$, which violates (H-4). This completes the proof of the theorem.

COROLLARY. *Let $\mathcal{F}$ be as in the theorem and suppose that* (H-1) *and* (H-3) *hold. Suppose that $\mathcal{F}$ has no critical points in $\overset{\circ}{E}$. Then either $\partial \mathcal{F}$ is cyclic* (*i.e., all isolated coverings possess cycles*) *or there exists $x \in \overset{\circ}{E}$ with* $\lim_{t \to \infty} d(\pi(x,t), \partial E) = 0$.

PROOF. If not, (H-2) and (H-4) both hold so the theorem holds. By uniform persistence and (H-1), $\Omega(\overset{\circ}{E})$ is a compact subset of $\overset{\circ}{E}$ which is a global weak attractor for $\overset{\circ}{E}$ (see [**1**]). By Theorem 2.8.6 of [**1**], $\mathcal{F}$ has a critical point in $\overset{\circ}{E}$, which is a contradiction, proving the corollary.

When $E = \mathbf{R}_+^n$ and the theorem is applied in the simplest instances, it says that for dissipative systems whose minimal sets have hyperbolic structure and whose boundary flow is acyclic, persistence implies uniform persistence (and hence the existence of an interior critical point). In most cases where persistence has been shown to hold (e.g. [**3**, **4**]) the hypotheses (H-1) to (H-4) are already required to expedite this demonstration, and now one can conclude the stronger assertion of uniform persistence.

Finally we note that the theorem can be applied in the more general context of a closed invariant subset K of E and conditions which establish that orbits in $E\backslash K$ are uniformly asymptotically bounded away from K, given that they are weakly asymptotically bounded away from K. We simply take $\mathcal{E} = E$ and define $\tilde{E} = E\backslash K$. Apply the theorem to the flow on $\tilde{E}$, so that (H-3) and (H-4) are applied to the restriction of the flow to K.

4. Discussion. It has been shown that under certain natural hypotheses, persistence implies uniform persistence. If any of these principal hypotheses are not satisfied, the conclusion of the theorem is not true, i.e. the system may persist without being uniformly persistent.

Examples to illustrate this may be found in the literature. If dissipativeness (H-1) does not hold, the Lotka-Volterra predator-prey system described in [**2**, Chapter 3] demonstrates persistence, but not uniform persistence. If (H-4) fails, then the nontransitive competition models given in [**13**, **15**] provide counterexamples to uniform persistence though persistence may occur. Finally, if the invariant sets are not isolated as described, i.e. (H-3) fails, the choice of parameters $\lambda_1 = \lambda_2$ in [**11**] gives a counterexample to uniform persistence, though persistence holds.

The situation for which E is not locally compact remains unresolved. It would be of interest in this case to know the circumstances under which persistence or uniform persistence occurs.

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