

ON TOPOLOGICALLY INDUCED GENERALIZED PROXIMITY RELATIONS

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1. Introduction. In this paper we examine one of the results in the theory of the proximity spaces of Efremovic [1]:

A set X with a binary relation " A close to B " (written $A \delta B$) is a proximity space if and only if there exists a compact Hausdorff space Y in which X can be topologically imbedded so that

(1.1) $A \delta B$ in X if and only if \overline{A} meets \overline{B} in Y

(\overline{A} denotes the closure of the set A) [2].

This proposition raises the question: Can we characterize the relations δ for which this result holds under weaker conditions on Y ? In §4 we give an affirmative answer (Theorem 5.3) using rather mild restrictions on Y and on the imbedding of X in Y . This result is essentially a corollary to a fundamental theorem (Theorem 4.2).

2. Symmetric generalized proximity spaces. As in [3] we define a *symmetric generalized proximity space* (or P_s -space) to be an abstract set X with a binary operation " $A \delta B$ " (a P_s -relation) on its power set satisfying the following axioms:

(P.1) $A \delta (B \cup C)$ implies that either $A \delta B$ or $A \delta C$.

(P.2) $A \delta B$ implies that A and B are nonvoid.

(P.3) If A meets B then $A \delta B$.

(P.4) $A \delta B$ and $b \delta C$ for all b in B implies that $A \delta C$.

(P.5) $A \delta B$ implies $B \delta A$.

We read the symbols " $A \delta B$ " as " A is close to B "; and we say that " A is remote from B " (in symbols, " A not δB ") if A is not close to B .

(2.1) The following facts are evident: (1) If $A \delta B$, $A \subset C$, and $B \subset D$, then $C \delta D$. (2) Define

$$A^\delta = \{x \in X: x \delta A\};$$

then in a P_s -space $(A^\delta) \delta (B^\delta)$ if and only if $A \delta B$.

(2.2) In [3] it is shown that there is a topology induced on every P_s -space (X, δ) by the closure operation $A \rightarrow A^\delta$. Moreover, this topology is *symmetric*: x in \bar{y} implies y in \bar{x} for all points $x, y \in X$. Clearly, every T_1 topological space is symmetric.

(2.3) **THEOREM.** *Given any symmetric topological space X define δ_0 by:*

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(2.4) $A \delta_0 B$ if and only if \overline{A} meets \overline{B} .

Then δ_0 is a P_s -relation and is compatible with the given topology: $x \delta_0 B$ if and only if $x \in \overline{B}$.

PROOF. We derive axioms (P.1) through (P.5) by use of the Kuratowski closure axioms [4]. Axioms (P.1), (P.2), (P.3) and (P.5) are trivial results of the closure axioms and (2.4). For (P.4), note that if for a point b and a set C we have $\overline{b} \cap \overline{C} \neq \emptyset$, then there exists a point c in C such that $c \in \overline{b}$. By symmetry then $b \in \overline{c} \subset \overline{C}$. Thus, if $\overline{A} \cap \overline{B} \neq \emptyset$ and $\overline{b} \cap \overline{C} \neq \emptyset$ for every b in B then $\overline{B} \subset \overline{C}$ and so $\overline{A} \cap \overline{C} \neq \emptyset$. It is now clear, from the above argument, that δ_0 is compatible with the given topology.

(2.5) THEOREM. Given a P_s -space (X, δ) and δ_0 defined by (2.4) in terms of the topology induced by δ we have that $A \delta_0 B$ implies that $A \delta B$ for all subsets A and B of X . Thus δ_0 is the smallest P -relation compatible with the topology in a symmetric topological space.

PROOF. The demonstration follows directly from (2.3), (2.1) and (P.3).

3. Clusters. A cluster π from a P_s -space (X, δ) is a class of subsets of X satisfying:

(C.1) $A \delta B$ for all $A, B \in \pi$.

(C.2) $A \cup B \in \pi$ implies that either $A \in \pi$ or $B \in \pi$.

(C.3) If $B \delta A$ for every A in π , then $B \in \pi$.

Note that this is the same definition used by Leader [5] in introducing clusters for Efremovic proximity spaces.

(3.1) THEOREM. For x , a point in a P_s -space (X, δ) , the class π_x of all subsets of X which are close to x is a cluster from X .

PROOF. We must show that π_x satisfies (C.1), (C.2) and (C.3). For (C.1) suppose $A, B \in \pi_x$. Then $x \delta A$ and $x \delta B$ so that, by (2.5), $A \delta B$. For (C.2) suppose $A \cup B \in \pi_x$. Then $x \delta (A \cup B)$ and, by (P.1), this means that either $x \delta A$ or $x \delta B$, that is, either $A \in \pi_x$ or $B \in \pi_x$. For (C.3) suppose that $A \delta C$ for every C in π_x . Since, by (P.3), $\{x\} \in \pi_x$, we have in particular, that $A \delta x$ or, $A \in \pi_x$.

(3.2) The following facts are easily established. (1) Any cluster π from a P_s -space (X, δ) is closed under the operation of supersets: if π is a cluster from X , $A \in \pi$, and $A \subset B$, then $B \in \pi$. (2) If $A \in \pi$, a cluster from X , and $a \delta B$ for every a in A , then $B \in \pi$. (3) If π and π' are clusters from X and π is a subclass of π' , then $\pi = \pi'$. (4) If a point x belongs to a cluster π , then π is just the class π_x of all subsets A of X such that $A \delta x$. (5) Given a cluster π from a nonvoid P_s -space

(X, δ) and any subset A of X , then either $A \in \pi$ or $X - A \in \pi$. (6) Let π be a cluster from (X, δ) . If A is a subset of X which meets every member of π , then $A \in \pi$.

4. Extensions characterized by clusters. We say that a subset X of a topological space Y is *regularly dense* in Y if and only if given U open in Y and p a point in U there exists a subset E of X with $p \in \bar{E} \subset U$, the closure being taken in Y .

(4.1) THEOREM. *If X is regularly dense in Y , then X is dense in Y . If Y is regular and X is dense in Y then X is regularly dense in Y .*

PROOF. Y is open in Y , hence for any point p in Y there exists a subset E of X such that $p \in \bar{E} \subset \bar{X} \subset Y$. Since this is true for any p in Y , we have $Y \subset \bar{X} \subset Y$.

For Y regular, $y \in Y$ and U an open set of Y containing y we have the existence of an open set V of Y containing y such that $\bar{V} \subset U$. Now $E = V \cap X$ is a subset of X and $\bar{E} = \text{Cl}(V \cap X) = \bar{V} \subset U$,¹ with the second equality following from the density of X in Y . Thus, $y \in \bar{E} \subset U$.

(4.2) THEOREM. *Given a set X and some binary relation δ on the power set of X , the following are equivalent:*

(I) *There exists a T_1 topological space Y and a mapping f of X into Y such that fX is regularly dense in Y and*

(4.3) *$A \delta B$ in X if and only if $\text{Cl}(fA)$ meets $\text{Cl}(fB)$ in Y .*

(II) *δ is a P_* -relation satisfying the additional axiom,*

(P.6) *Given $A \delta B$ in X there exists a cluster π to which both A and B belong.*

PROOF. Suppose that (I) holds and define δ by (4.3). (P.1), (P.2), (P.3) and (P.5) are trivial consequences of the properties of closure. For (P.4) suppose that $A \delta B$ and $b \delta C$ for all b in B . Then $\text{Cl}(fA) \cap \text{Cl}(fB) \neq \emptyset$ and $\text{Cl}(fb) \cap \text{Cl}(fC) \neq \emptyset$ for all b in B , which since Y is T_1 , implies that $fb \in \text{Cl}(fC)$ for all b in B . Thus $fB \subset \text{Cl}(fC)$ or $\text{Cl}(fB) \subset \text{Cl}(fC)$ so that $\text{Cl}(fA) \cap \text{Cl}(fC) \neq \emptyset$ showing that $A \delta C$. For (P.6), since $\text{Cl}(fA) \cap \text{Cl}(fB) \neq \emptyset$, let $c \in \text{Cl}(fA) \cap \text{Cl}(fB)$ and define π to be the class of all subsets S of X such that $c \in \text{Cl}(fS)$. Clearly A and B are in π and in showing that π is a cluster the demonstrations of (C.1) and (C.2) are trivial. For (C.3) suppose that $\text{Cl}(fD) \cap \text{Cl}(fC) \neq \emptyset$ for every C in π but that $D \notin \pi$, i.e., $c \notin \text{Cl}(fD)$. Thus, $c \in Y - \text{Cl}(fD)$ and since fX is regularly dense in Y there exists a subset E of X such that $c \in \text{Cl}(fE) \subset Y - \text{Cl}(fD)$. That is, there

¹ Where Cl stands for closure.

exists an E in π such that $\text{Cl}(fD) \cap \text{Cl}(fE) = \emptyset$. This contradicts the hypothesis of (C.3). Thus (II) is satisfied.

For the converse suppose that (II) holds. Given x in X the class π_x of all subsets A of X such that $x \delta A$ is a cluster from X , by (3.1). Thus for any subset A of X , let α be the set of all clusters π_a determined by the points a in A . Let $\bar{\alpha}$ be the set of all clusters to which A belongs. By (P.3), $A \in \pi_a$ for each a in A and so $\alpha \subset \bar{\alpha}$. We will denote \mathfrak{X} , the set of all clusters from X , by Y .

A subset A of X absorbs a subset β of Y if and only if A belongs to every cluster in β , that is, if and only if $\bar{\alpha}$ contains β . For any subset β of Y we define the closure, $\text{cl}(\beta)$, of β by

(4.4) $\pi \in \text{cl}(\beta)$ if and only if every subset E of X which absorbs β is in π .

We next show that

(4.5) $\text{cl}(\alpha) = \bar{\alpha}$.

For if $\pi \in \text{cl}(\alpha)$ then since A absorbs α , $A \in \pi$ so that $\pi \in \bar{\alpha}$. On the other hand, if $\pi \in \bar{\alpha}$ then $A \in \pi$. Now let P be in every π_a in α , i.e., $P \delta a$ for every a in A and hence $A \subset P^\delta$. Thus, by (3.2), (2), $P \in \pi$ so that $\pi \in \text{cl}(\alpha)$.

We now show that the Kuratowski closure axioms are satisfied by the closure defined by (4.4).

(K.1) $\beta \subset \text{cl}(\beta)$: This is trivial since if E absorbs β then $E \in \pi$ for every $\pi \in \beta$.

(K.2) $\text{cl}(\emptyset) = \emptyset$: Suppose $\pi \in \text{cl}(\emptyset)$. Since it is vacuously true that every subset of X absorbs \emptyset , we then have that every subset of X is in π . In particular, \emptyset and X are in π . Thus, $\emptyset \delta X$, by (C.1), contradicting (P.2).

(K.3) $\text{cl}(\text{cl}(\beta)) \subset \text{cl}(\beta)$: Suppose $\pi \in \text{cl}(\text{cl}(\beta))$ and that E absorbs β . By (4.4), E absorbing β implies that E absorbs $\text{cl}(\beta)$. Hence $E \in \pi$ showing that $\pi \in \text{cl}(\beta)$.

(K.4) $\text{cl}(\beta \cup \beta') = \text{cl}(\beta) \cup \text{cl}(\beta')$: Suppose that $\pi \in \text{cl}(\beta \cup \beta')$ and that A absorbs β and A' absorbs β' . Then, by (3.2), (1), $A \cup A'$ absorbs $\beta \cup \beta'$ so that $A \cup A' \in \pi$. But, by (C.2), this means that either $A \in \pi$ or $A' \in \pi$, that is $\pi \in \text{cl}(\beta)$ or $\pi \in \text{cl}(\beta')$. Thus $\pi \in \text{cl}(\beta) \cup \text{cl}(\beta')$ and we have $\text{cl}(\beta \cup \beta') \subset \text{cl}(\beta) \cup \text{cl}(\beta')$. On the other hand, $\pi \in \text{cl}(\beta) \cup \text{cl}(\beta')$ implies that either $\pi \in \text{cl}(\beta)$ or $\pi \in \text{cl}(\beta')$. Now if E absorbs $\beta \cup \beta'$, then E absorbs β and also absorbs β' . Hence, $E \in \pi$ showing that $\pi \in \text{cl}(\beta \cup \beta')$ and (K.4) holds.

To show that the topology is T_1 , suppose $\pi' \in \text{cl}(\pi)$, where π and π' are clusters from X . This means that every set in π is also in π' . Thus, $\pi \subset \pi'$ and by (3.2), (3), $\pi = \pi'$. Hence, $\text{cl}(\pi) = \pi$ for every point π in the space Y .

Now the correspondence which assigns to each point x in X the cluster π_x determined by it is a well-defined transformation mapping X into Y which we will denote by f . Note that $fA = \bar{\alpha}$ for every subset A of X , so in order to show that (4.3) holds we must show that, using (4.5),

$$(4.6) \quad A \delta B \text{ in } X \text{ if and only if } \bar{\alpha} \text{ meets } \bar{\beta} \text{ in } Y.$$

So if $A \delta B$ there exists, by (P.6), a cluster π to which both A and B belong. Thus, by definition of $\bar{\alpha}$, we have $\pi \in \bar{\alpha} \cap \bar{\beta}$. On the other hand, if $\pi \in \bar{\alpha} \cap \bar{\beta}$ then A and B are in π so that, by (C.1), $A \delta B$.

To show that $fX = \mathfrak{X}$ is regularly dense in Y suppose that α is an open subset of Y and that $\pi \in \alpha$. We thus have $\pi \in Y - \alpha = \text{cl}(Y - \alpha)$. This means, by (4.4), that there exists some subset E of X such that E is in every cluster of $Y - \alpha$ but that E is not in π . Hence, by (C.3), there is a C in π such that E not δ C .

Since \bar{c} is the set of all clusters to which C belongs we have $\pi \in \bar{c}$. And since E belongs to every cluster in $Y - \alpha$ and E not δ C , then C cannot belong to any cluster in $Y - \alpha$, by (C.1). Hence \bar{c} is contained in α and we have shown that \mathfrak{X} is regularly dense in Y .

The proof is now complete.

5. Symmetric P_1 -spaces. A P_s -space (X, δ) in which δ satisfies the additional axiom

$$(5.1) \quad x \delta y \text{ implies } x = y \text{ for all points } x, y \in X$$

is called a *symmetric P_1 -space* (see [3]). The following theorem follows directly from (C.1) and (5.1).

(5.2) THEOREM. *Every cluster π from a symmetric P_1 -space (X, δ) possesses at most one point.*

(5.3) THEOREM. *Given a set X and some binary relation δ on the power set of X , the following are equivalent:*

(I') *There exists a T_1 topological space Y in which X can be topologically imbedded as a regularly dense subset so that (1.1) holds.*

(II') *δ is a symmetric P_1 -relation satisfying (P.6).*

PROOF. The demonstration is similar to that of Theorem (4.2). To see that (5.1) holds, note that $\bar{x} \cap \bar{y} \neq \emptyset$ implies that $x \cap y \neq \emptyset$, or $x = y$.

To show that our imbedding is topological we note first that, because of (5.2) the correspondence between X and \mathfrak{X} induced by the identification of x with the cluster π_x determined by it is one-to-one. To see that the mapping is bicontinuous we must show that if A is a

subset of X , $x \in A^\delta$ if and only if $\pi_x \in \text{kl}(\mathcal{A})$, where $\text{kl}(\mathcal{A})$ is the closure of \mathcal{A} in \mathfrak{X} relative to the space Y .

So suppose $x \in A^\delta$ and that P absorbs \mathcal{A} . Then P is a member of every π_a in \mathcal{A} and it follows that $a \delta P$ for every a in A . Thus, $A \subset P^\delta$ and since $A \in \pi_x$ we have, from (3.2), (2), that $P \in \pi_x$. Thus, $\pi_x \in \text{kl}(\mathcal{A})$.

On the other hand, suppose $\pi_x \in \text{kl}(\mathcal{A})$. Then since A absorbs \mathcal{A} we have $A \in \pi_x$, i.e., $A \delta x$ and hence $x \in A^\delta$. This completes the proof.

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