THE RADIUS OF UNIVALENCE OF CERTAIN ANALYTIC FUNCTIONS

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1. **Introduction.** Suppose that $f(z)=z+a_2z^2+\cdots$ is analytic for $|z|<1$. If $\operatorname{Re}\{f(z)/z\}>0$ for $|z|<1$ then $f(z)$ is univalent in $|z|<\sqrt{2}-1$ [**5**, Theorem 3; **7**]. The function $f(z)=(z+z^2)/(1-z)$ satisfies the hypotheses but is univalent in no circle $|z|<r$ for $r>\sqrt{2}-1$ since its derivative vanishes at $z=\sqrt{2}-1$.

In this paper we generalize the above theorem for functions whose power series begins $f(z)=z+a_{n+1}z^{n+1}+\cdots$. The estimate used to obtain this result is further used to find the radius of convexity for functions $f(z)=z+a_{n+1}z^{n+1}+\cdots$ which are analytic and satisfy $\operatorname{Re} f'(z)>0$ for $|z|<1$. For $n=1$ this theorem is not new [**5**, Theorem 2; **10**, p. 284]. The condition $\operatorname{Re} f'(z)>0$ is known to be sufficient for the univalency of $f(z)$ in $|z|<1$ [**1**, p. 18].

We consider the problem of finding the radius of univalence for functions $f(z)=z+a_2z^2+\cdots$ which are analytic and satisfy $\operatorname{Re}\{f(z)/g(z)\}>0$ for $|z|<1$, where $g(z)=z+b_2z^2+\cdots$ is analytic and univalent for $|z|<1$. In the case that $g(z)$ is either starlike or convex this problem is solved. We take particular advantage of the condition $\operatorname{Re}\{zf'(z)/f(z)\}>0$ for $|z|<r$, which is necessary and sufficient for $f(z)$ to be univalent and starlike in $|z|<r$ [**8**, p. 105, problem 109]. For arbitrary univalent functions $g(z)$ we only obtain an estimate for the radius of univalence for $f(z)$.

2. Lemma 1. *Suppose that* $h(z)=1+c_nz^n+\cdots$ *is analytic and satisfies* $\operatorname{Re} h(z)>0$ *for* $|z|<1$. *Then*

$$\left|\frac{h'(z)}{h(z)}\right| \leqq \frac{2n|z|^{n-1}}{1-|z|^{2n}}.$$

Proof. Let $k(z)=(1-h(z))/(1+h(z))=d_nz^n+\cdots$. Then $k(z)$ is analytic for $|z|<1$ and $|k(z)|<1$. Thus, $k(z)=z^n\phi(z)$ where $\phi(z)$ is analytic for $|z|<1$ and $|\phi(z)|\leqq 1$. For such functions we have

$$|\phi'(z)| \leqq \frac{1-|\phi(z)|^2}{1-|z|^2} \tag{1}$$

[**2**, p. 18].

Expressing $h(z)$ and $h'(z)$ in terms of $\phi(z)$ gives

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$$\frac{h'(z)}{h(z)} = -2z^{n-1}\frac{z\phi'(z)+n\phi(z)}{1-z^{2n}\phi^2(z)}$$

$$\left|\frac{h'(z)}{h(z)}\right| \leqq 2|z|^{n-1}\frac{|z|\,|\phi'(z)|+n|\phi(z)|}{1-|z|^{2n}|\phi(z)|^2}.$$

Using (1) we obtain

$$\left|\frac{h'(z)}{h(z)}\right| \leqq \frac{2|z|^{n-1}}{1-|z|^2}\,\frac{|z|(1-|\phi(z)|^2)+n(1-|z|^2)|\phi(z)|}{1-|z|^{2n}|\phi(z)|^2}.$$

[1]To prove the lemma it is sufficient to show that for $|z|=r$, $0<r<1$,

$$\frac{r(1-|\phi(z)|^2)+n(1-r^2)|\phi(z)|}{1-r^{2n}|\phi(z)|^2} \leqq \frac{n(1-r^2)}{1-r^{2n}}.$$

Letting $x=|\phi(z)|$ this is equivalent to $(1-x)F_n(x)\geqq 0$ for $0\leqq x\leqq 1$, where

$$F_n(x) = a-bx,\quad a = n(1-r^2)-r(1-r^{2n}) > (1-r^2)(n-nr) > 0,$$

$$\begin{aligned} b &= r(1-r^{2n})-nr^{2n}(1-r^2)\\ &= r(1-r^2)(1+r^2+r^4+\cdots+r^{2n-2}-nr^{2n-1})\\ &= r(1-r^2)\{(1-r^{2n-1})+(r^2-r^{2n-1})+\cdots+(r^{2n-2}-r^{2n-1})\}\\ &> 0. \end{aligned}$$

Since $F_n(x)\geqq F_n(1)$ we can prove $F_n(x)\geqq 0$ by showing that $F_{n+1}(1)\geqq F_n(1)$ and $F_1(1)\geqq 0$.

$$\begin{aligned} &F_{n+1}(1)-F_n(1)\\ &= (1-r^2)(1-r^{2n+1}-r^{2n+1}+r^{2n+2}-nr^{2n}+nr^{2n+2})\\ &= (1-r^2)(1-r)\{1+r+r^2+\cdots+r^{2n}-r^{2n+1}-nr^{2n}-nr^{2n+1}\}\\ &> 0. \end{aligned}$$

This inequality follows since the negative terms in the brackets can be expressed as $2n+1$ terms each of which is numerically less than a corresponding positive term.

Finally, $F_1(1)=(1+r)(1-r)^3>0$.

One can show that the equality holds in the lemma only for the functions $h(z)=(1-\epsilon z^n)/(1+\epsilon z^n)$ where $|\epsilon|=1$ and for appropriate values of z.

THEOREM 1. *Suppose that $f(z)=z+a_{n+1}z^{n+1}+\cdots$ is analytic and*

[1] I would like to thank the referee of this paper for simplifying my argument for the remaining part of the proof.

satisfies $\operatorname{Re}\{f(z)/z\}>0$ *for* $|z|<1$. *Then* $f(z)$ *is univalent and starlike in* $|z|<((n^2+1)^{1/2}-n)^{1/n}$.

Proof. Since $\operatorname{Re}\{f(z)/z\}>0$ we can infer that $f(z)$ cannot vanish in $|z|<1$ except for a simple zero at $z=0$. Let

$$h(z)=\frac{f(z)}{z}=1+a_{n+1}z^n+\cdots,\ \operatorname{Re} h(z)>0 \qquad \text{for } |z|<1.$$

From Lemma 1 we have

$$\left|\frac{zh'(z)}{h(z)}\right| \leqq \frac{2n|z|^n}{1-|z|^{2n}}.$$

Also

$$\frac{zf'(z)}{f(z)}=1+\frac{zh'(z)}{h(z)}.$$

Therefore, $f(z)$ will be univalent and starlike if $|zh'(z)/h(z)|<1$. From the above estimate this is satisfied if $(2n|z|^n)/(1-|z|^{2n})<1$, i.e., for $|z|<((n^2+1)^{1/2}-n)^{1/n}$.

The function $f(z)=(z+z^{n+1})/(1-z^n)=z+2z^{n+1}+\cdots$ satisfies

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\}>0 \qquad \text{for } |z|<1$$

but is not univalent in $|z|<r$ for $r>r_n=((n^2+1)^{1/2}-n)^{1/n}$ since $f'(r_ne^{i(\pi/n)})=0$.

Theorem 2. *Suppose that* $f(z)=z+a_{n+1}z^{n+1}+\cdots$ *is analytic and satisfies* $\operatorname{Re} f'(z)>0$ *for* $|z|<1$. *Then* $f(z)$ *is convex in* $|z|<((n^2+1)^{1/2}-n)^{1/n}$.

Proof. We can apply Lemma 1 to $f'(z)=1+(n+1)a_{n+1}z^n+\cdots$ since $\operatorname{Re} f'(z)>0$. This gives $|f''(z)/f'(z)|\leqq 2n|z|^{n-1}/1-|z|^{2n}$. The condition $\operatorname{Re}\{(zf''(z)/f'(z))+1\}>0$ for $|z|<r$ is necessary and sufficient for $f(z)$ to map $|z|<r$ onto a convex domain [8, problem 108, p. 105]. This condition is satisfied if $|zf''(z)/f'(z)|<1$. From the above estimate we can deduce that $f(z)$ is convex if $(2n|z|^n)/(1-|z|^{2n})<1$. This inequality is equivalent to $|z|<((n^2+1)^{1/2}-n)^{1/n}$.

The function

$$f(z)=\int_0^z \frac{1+\sigma^n}{1-\sigma^n}\,d\sigma=z+\frac{2}{n+1}z^{n+1}+\cdots$$

is an extremal function for Theorem 2.

3. THEOREM 3. *Suppose that* $f(z)=z+a_2z^2+\cdots$ *and* $g(z)=z+b_2z^2+\cdots$ *are analytic for* $|z|<1$ *and* $g(z)$ *is univalent and starlike for* $|z|<1$. *If* $\operatorname{Re}\{f(z)/g(z)\}>0$ *for* $|z|<1$ *then* $f(z)$ *is univalent and starlike in* $|z|<2-\sqrt{3}$.

PROOF. The hypotheses imply that $f(z)$ and $g(z)$ do not vanish in $|z|<1$ except for the simple zero at $z=0$. Let

$$h(z)=\frac{f(z)}{g(z)}=1+c_1z+\cdots,\ \operatorname{Re} h(z)>0 \qquad \text{for } |z|<1.$$

Applying Lemma 1 to $h(z)$ for $n=1$ gives $|zh'(z)/h(z)|\leqq 2|z|/(1-|z|^2)$. Since $g(z)$ is starlike $\operatorname{Re}\{zg'(z)/g(z)\}>0$ for $|z|<1$. Thus $\operatorname{Re}\{zg'(z)/g(z)\}\geqq(1-|z|)/(1+|z|)$ [8, problem 287, p. 140].

$$\frac{zf'(z)}{f(z)}=\frac{zg'(z)}{g(z)}+\frac{zh'(z)}{h(z)}$$

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}\geqq\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\}-\left|\frac{zh'(z)}{h(z)}\right|\geqq\frac{1-|z|}{1+|z|}-\frac{2|z|}{1-|z|^2}$$
$$=\frac{1-4|z|+|z|^2}{1-|z|^2}.$$

Thus, $\operatorname{Re}\{zf'(z)/f(z)\}>0$ if $1-4|z|+|z|^2>0$. The last inequality is satisfied for $|z|<2-\sqrt{3}$. Therefore $f(z)$ is univalent in $|z|<2-\sqrt{3}$ and maps that circle onto a starlike domain.

The function $f(z)=(z+z^2)/(1-z)^3$ satisfies the hypotheses of Theorem 3 where $g(z)=z/(1-z)^2$ and $h(z)=(1+z)/(1-z)$. The derivative of this function vanishes at $z=\sqrt{3}-2$. Thus, it is univalent in no circle $|z|<r$ with $r>2-\sqrt{3}$.

For a part of the next theorem we need a sharpening of Lemma 1 for $n=1$. This result is known but we give a short proof of it here.

LEMMA 2. *Suppose that* $h(z)=1+c_1z+\cdots$ *is analytic and satisfies* $\operatorname{Re} h(z)>0$ *for* $|z|<1$. *Then* $|h'(z)|\leqq 2\operatorname{Re} h(z)/(1-|z|^2)$.

PROOF.[2] Let $\phi(z)=(1-h(z))/(1+h(z))$, $|\phi(z)|<1$ for $|z|<1$. Using

$$(1) \qquad |\phi'(z)|\leqq\frac{1-|\phi(z)|^2}{1-|z|^2}$$

gives

[2] The author thanks the referee for indicating that the proof of this lemma can be obtained so readily from the estimate (1).

$$|h'(z)| \leqq \frac{|1+h(z)|^2 - |1-h(z)|^2}{2(1-|z|^2)}.$$

The lemma follows by noting that $|1+h(z)|^2-|1-h(z)|^2=4 \operatorname{Re} h(z)$.

THEOREM 4. *Suppose that* $f(z) = z + a_2z^2 + \cdots$ *and* $g(z) = z + b_2z^2 + \cdots$ *are analytic for* $|z| < 1$ *and* $g(z)$ *is univalent and convex for* $|z| < 1$. *If* $\operatorname{Re}\{f(z)/g(z)\} > 0$ *for* $|z| < 1$ *then* $\operatorname{Re}\{f'(z)/g'(z)\} > 0$ *for* $|z| < \frac{1}{3}$. *Also,* $f(z)$ *is univalent and starlike for* $|z| < \frac{1}{3}$.

PROOF. The hypotheses imply that $f(z)$, $g(z)$ and $g'(z)$ do not vanish in $|z| < 1$ except for the simple zeros of $f(z)$ and $g(z)$ at $z=0$. Let $h(z) = f(z)/g(z) = 1 + c_1z + \cdots$, $\operatorname{Re} h(z) > 0$ for $|z| < 1$.

Applying Lemma 2 to $h(z)$ gives $|h'(z)| \leqq 2 \operatorname{Re} h(z)/(1-|z|^2)$. Since $g(z)$ is univalent and convex for $|z| < 1$ we have $\operatorname{Re}\{zg'(z)/g(z)\} > \frac{1}{2}$ for $|z| < 1$ and consequently $\operatorname{Re}\{zg'(z)/g(z)\} \geqq (1+|z|)^{-1}$ [**6**; **9**]. This implies $|g(z)/g'(z)| \leqq |z|(1+|z|)$.

$$\frac{f'(z)}{g'(z)} = h(z) + \frac{g(z)}{g'(z)} h'(z)$$

$$\begin{aligned} \operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} &\geqq \operatorname{Re} h(z) - \left|\frac{g(z)}{g'(z)} h'(z)\right| \\ &\geqq \operatorname{Re} h(z) - |z|(1+|z|)\frac{2 \operatorname{Re} h(z)}{1-|z|^2} \\ &= \frac{1-3|z|}{1-|z|} \operatorname{Re} h(z). \end{aligned}$$

Thus, for $|z| < \frac{1}{3}$ $\operatorname{Re}\{f'(z)/g'(z)\} > 0$. This shows that $f(z)$ is univalent and close-to-convex for $|z| < \frac{1}{3}$ [**4**].

Let us show that $f(z)$ maps $|z| < \frac{1}{3}$ onto a starlike domain.

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)}$$

$$\begin{aligned} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} &\geqq \operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} - \left|\frac{zh'(z)}{h(z)}\right| \\ &\geqq \frac{1}{1+|z|} - \frac{2|z|}{1-|z|^2} = \frac{1-3|z|}{1-|z|^2}. \end{aligned}$$

For $|z| < \frac{1}{3}$ $\operatorname{Re}\{zf'(z)/f(z)\} > 0$. Thus, $f(z)$ is starlike in $|z| < \frac{1}{3}$.

Theorem 4 gives the radius of univalence for the class of functions considered. In order to show this let $f(z) = (z+z^2)/(1-z)^2$, $g(z)$

$=z/(1-z)$. Then, $g(z)$ is univalent and convex for $|z|<1$. Here, $h(z)=(1+z)/(1-z)$ and therefore $\operatorname{Re} h(z)>0$. This function $f(z)$ is univalent in no circle $|z|<r$ with $r>\frac{1}{3}$ since $f'(-\frac{1}{3})=0$.

THEOREM 5. *Suppose that* $f(z)=z+a_2z^2+\cdots$ *and* $g(z)=z+b_2z^2+\cdots$ *are analytic for* $|z|<1$ *and* $g(z)$ *is univalent in* $|z|<1$. *If* $\operatorname{Re}\{f(z)/g(z)\}>0$ *for* $|z|<1$ *then* $f(z)$ *is univalent in* $|z|<1/5$.

PROOF. Let $h(z)=f(z)/g(z)=1+c_1z+\cdots$, $\operatorname{Re} h(z)>0$ for $|z|<1$.

To show that $f(z)$ is univalent in $|z|\leqq r$ it suffices to show that $f(z)$ is univalent on $|z|=r$. Let $z_1\neq z_2$, $|z_1|=|z_2|=r$. Then, $f(z_1)=f(z_2)$ can be written

$$\frac{1}{g(z_1)}\,\frac{g(z_2)-g(z_1)}{z_2-z_1}=-\frac{1}{h(z_2)}\,\frac{h(z_2)-h(z_1)}{z_2-z_1}.$$

Thus, if

$$\left|\frac{g(z_2)-g(z_1)}{g(z_1)(z_2-z_1)}\right|>\left|\frac{h(z_2)-h(z_1)}{h(z_2)(z_2-z_1)}\right|$$

then $f(z)$ is univalent in $|z|\leqq r$.

Let $k(z)=(1-h(z))/(1+h(z))$, $|k(z)|<1$ for $|z|<1$ and $k(0)=0$. Therefore $|k'(z)|\leqq 1$ for $|z|\leqq\sqrt{2}-1$ [2, p. 19]. From the representation $k(z_2)-k(z_1)=\int_{z_1}^{z_2}k'(z)dz$ where the path of integration is the line segment from z_1 to z_2 the estimate on $k'(z)$ gives $|(k(z_2)-k(z_1))/(z_2-z_1)|\leqq 1$ for $r\leqq\sqrt{2}-1$. Expressing $h(z)$ in terms of $k(z)$ yields

$$\frac{h(z_2)-h(z_1)}{h(z_2)(z_2-z_1)}=-2\,\frac{k(z_2)-k(z_1)}{z_2-z_1}\quad\frac{1}{(1+k(z_1))(1-k(z_2))}$$

$$\left|\frac{h(z_2)-h(z_1)}{h(z_2)(z_2-z_1)}\right|\leqq\frac{2}{(1-|k(z_1)|)(1-|k(z_2)|)}$$

$$\leqq\frac{2}{(1-|z_1|)(1-|z_2|)}=\frac{2}{(1-r)^2}.$$

Here we have used Schwarz's lemma $|k(z)|\leqq|z|$.

Since $g(z)=z+b_2z^2+\cdots$ is analytic and univalent for $|z|<1$

$$\left|\frac{g(z_2)-g(z_1)}{z_2-z_1}\right|\geqq|g(z_1)g(z_2)|\,\frac{1-r^2}{r^2}$$

[3]. Using this estimate and the distortion theorem $|g(z)|\geqq|z|/(1+|z|)^2$ we obtain

$$\left|\frac{g(z_2) - g(z_1)}{g(z_1)(z_2 - z_1)}\right| \geqq \frac{1 - r}{r(1 + r)}.$$

Therefore, $f(z)$ is univalent in $|z| \leqq r$ if $r \leqq \sqrt{2} - 1$ and $(1-r)/(r(1+r)) > 2/(1-r)^2$. The last inequality is equivalent to $1-5r+r^2-r^3>0$. Since the equation $1-5r+r^2-r^3=0$ has one positive root r_0, where $0.20<r_0<0.21$, we can infer that $f(z)$ is univalent in $|z|<r_0$. In particular, $f(z)$ is univalent in $|z|<1/5$.

The circle $|z|<r_0$ is not the circle of univalence for the functions $f(z)$ which satisfy the hypotheses of Theorem 5. If it were then we must have $|g(z)| = |z|/(1+|z|)^2$ for some z. This estimate holds only for the functions $g(z) = z/(1+\epsilon z)^2$ where $|\epsilon|=1$. Since these functions are starlike for $|z|<1$ Theorem 3 implies that $f(z)$ would be univalent $|z|<2-\sqrt{3}$. However, $2-\sqrt{3}=0.267 \cdots > r_0$.

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