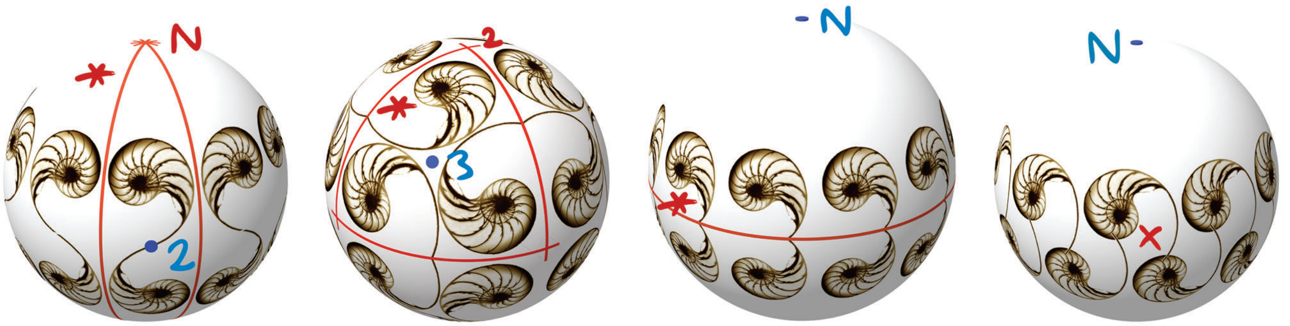
Conway's Mathematics After Conway



Alex Ryba with papers contributed by Philip Ehrlich, Richard Kenyon, Jeffrey Lagarias, James Propp, and Louis H. Kauffman

Mathematicians always queued to hear John Conway speak and delved into his writing. They expected to be entertained by beautiful mathematics and believed that they would emerge with valuable enlightenment. John welcomed this attention and considered it his duty to make his mathematics elegant. What really made his mathematics valuable was his wealth of insight. Wherever he worked, he opened up avenues for us to follow.

John challenged and inspired us in many different ways. His Game of Life has been investigated by thousands, while his *river* method provides a novel approach to the classical subject of quadratic forms. His *orbifold notation* was devised in collaboration with Bill Thurston. It gives a new way to describe symmetry patterns like those in the opening images above. The images are taken from the beautiful book, *The Symmetries of Things*, coauthored by John, Heidi Burgiel, and Chaim Goodman-Strauss. Group theorists, like me, are drawn to his paper "Monstrous Moonshine," written with his friend Simon Norton. John and Simon filled the paper with what were outlandish examples and provocative conjectures about the Monster group. Mathematicians immediately realized that the group was much more than a sporadic curiosity, as yet unconstructed. Richard Borcherds created his new theory of Vertex Algebras to prove the conjectures in John's paper. McKay and Sebbar aptly call Moonshine "21st century mathematics in the 20th century" [15]. John's influence can be seen across so many areas of mathematics that a complete account would fill many volumes, which would continue to expand forever. The three papers that follow present very different parts of mathematics where modern approaches are built on foundations laid by John.

The theory of surreal numbers was invented by John in the early 1970s. He expected this to become his most influential creation and was convinced that the theory would continue to evolve long into the future. Philip Ehrlich explains how this theory has progressed in the half century since its introduction.

John's algebraic approach to certain tiling problems is explained by Richard Kenyon, Jeffrey Lagarias, and James Propp. These tiling problems are mathematical questions. However, they used to be viewed as recreational because each required its own special trick. John's novel and very general approach showed that the problems belong to the field of Combinatorial Group Theory. John's vision across the whole breadth of mathematics allowed him to make many similar unexpected connections between fields.

Alex Ryba is a professor of computer science at Queens College, CUNY. His email address is `Alexander.Ryba@qc.cuny.edu`.

Communicated by Notices *Associate Editor Steven Sam.*

For permission to reprint this article, please contact: `reprint-permission@ams.org`.

DOI: https://doi.org/10.1090/noti2513

Knot theory was John's first research interest in mathematics. He was still in high school when he started to develop his Skein Theory. Louis Kauffman explains that the monumental 1983 paper of Vaughan Jones showed that the Jones polynomial satisfies a Conway type skein relation. A number of mathematicians were inspired to extend the Conway skein ideas to produce the Homflypt and Kauffman polynomials. This led directly to explosive progress in modern Knot Theory.

John saw himself as a link in a chain that stretched back through the mathematicians who created the fields where he worked and played. He particularly admired Georg Cantor, whose ordinals were uppermost in his mind when he worked on the surreal numbers. John, in turn, laid out new areas for others to cultivate after him. His legacy will be owned and continued by the mathematicians for whom he sowed so many seeds.

The Surreal Numbers and Their Aftermath

Philip Ehrlich

In 1970, as an outgrowth of his work on combinatorial games, J. H. Conway introduced a real-closed field, dubbed **No**, containing the reals and the ordinals, the arithmetic of the latter being the natural sums and products due to Hessenberg and Hausdorff rather than the usual non-commutative, non-associative sums and products of Cantor. Being a real-closed field containing the reals and the ordinals, **No** also contains a great many less familiar numbers including $-\omega$, $\omega/2$, $1/\omega$, $\sqrt{\omega}$ and $\omega-\pi$ to name only a few, where ω is the least infinite ordinal. Indeed, as Conway aptly quips, this particular real-closed field is so remarkably inclusive that it may be said to contain "All Numbers Great and Small" ([5, p. 3]). In this regard, **No** bears much the same relation to ordered fields that the ordered field $\mathbb{R}$ of real numbers bears to Archimedean ordered fields. Following D. E. Knuth, the members of **No** have come to be called *surreal numbers*.

While each surreal is an ordinary set, **No** is not. To address this, Conway formalizes his theory in NBG (*Von Neumann-Bernays-Gödel set theory*). Unlike standard *Zermelo-Fraenkel set theory* (ZF), whose sole entities are sets, NBG contains classes that are sets, as well as classes such as **No** and the class **On** of ordinals that are larger than any set, called *proper classes*.

Philip Ehrlich is a professor emeritus of philosophy at the Ohio University. His email address is ehrlich@ohio.edu.

Against this set-theoretic backdrop, the relation between the inclusiveness of $\mathbb{R}$ and **No** may be made precise by the following result collectively due to Conway [5, pp. 42-43] and Ehrlich [7]: *whereas* $\mathbb{R}$ *is (up to isomorphism) the unique universally embedding Archimedean ordered field,* **No** *is (up to isomorphism) the unique universally embedding ordered field*, where an ordered field (Archimedean ordered field) A is said to be *universally embedding* if for each ordered subfield (Archimedean ordered subfield) B of A, whose universe is a set, and every ordered field (Archimedean ordered field) B' extending B, there is an embedding $f : B' \to A$ that is the identity on B. Thus, starting with a categorical characterization of the inclusiveness of $\mathbb{R}$ that makes use of the Archimedean axiom, one obtains a categorical characterization of the inclusiveness of **No** by simply deleting the Archimedean condition. On the basis of this observation, that Conway initially thought "too good to be true," Ehrlich (e.g., [7, 9]) has suggested that whereas $\mathbb{R}$ should merely be regarded as the *arithmetic continuum* (modulo the Archimedean axiom), **No** may be regarded as the *absolute arithmetic continuum* (modulo NBG).

Like the ordered set of reals, the ordered class of surreals may be constructed in a variety of ways (e.g., [2, 5, 9, 11]). In Conway's construction, which generalizes aspects of Dedekind's cut construction of $\mathbb{R}$ and von Neumann's construction of **On**, the members of **No** (or, more property speaking, *their vast array of equivalent representations*) are extracted from an antecedently, inductively defined partially ordered class of *games* vis-à-vis the following inductive definition.

Construction of Surreal Numbers

If L and R are two sets of surreal numbers such that no member of L is greater than or equal to any member of R, then there is a surreal number $\{L|R\}$. All surreal numbers are constructed in this way.

In accordance with his convention for games, which likewise are equivalence classes of representatives of the form $\{L|R\}$, Conway denotes each surreal $x = \{L|R\}$ by '$x = \{x^L|x^R\}$' where the x^L's and x^R's—the *left* and *right options* of x—are understood to range over the members of L and the members of R, respectively. Using this convention, Conway's construction of the surreal numbers, which is carried out in stages (called "days") indexed over the ordinals, can be informally described as follows.

On the 0th day, beginning with the empty set $\emptyset$ (of surreal numbers), Conway constructs the surreal number

$$0 = \{|\};$$

and on the 1st day, the surreal numbers

$$-1 = \{|0\} \qquad 1 = \{0|\};$$

and then on the 2nd day, the surreal numbers

$$-2 = \{|0,1\} \; -1/2 = \{-1|0\} \; 1/2 = \{0|1\} \; 2 = \{0,1|\},$$

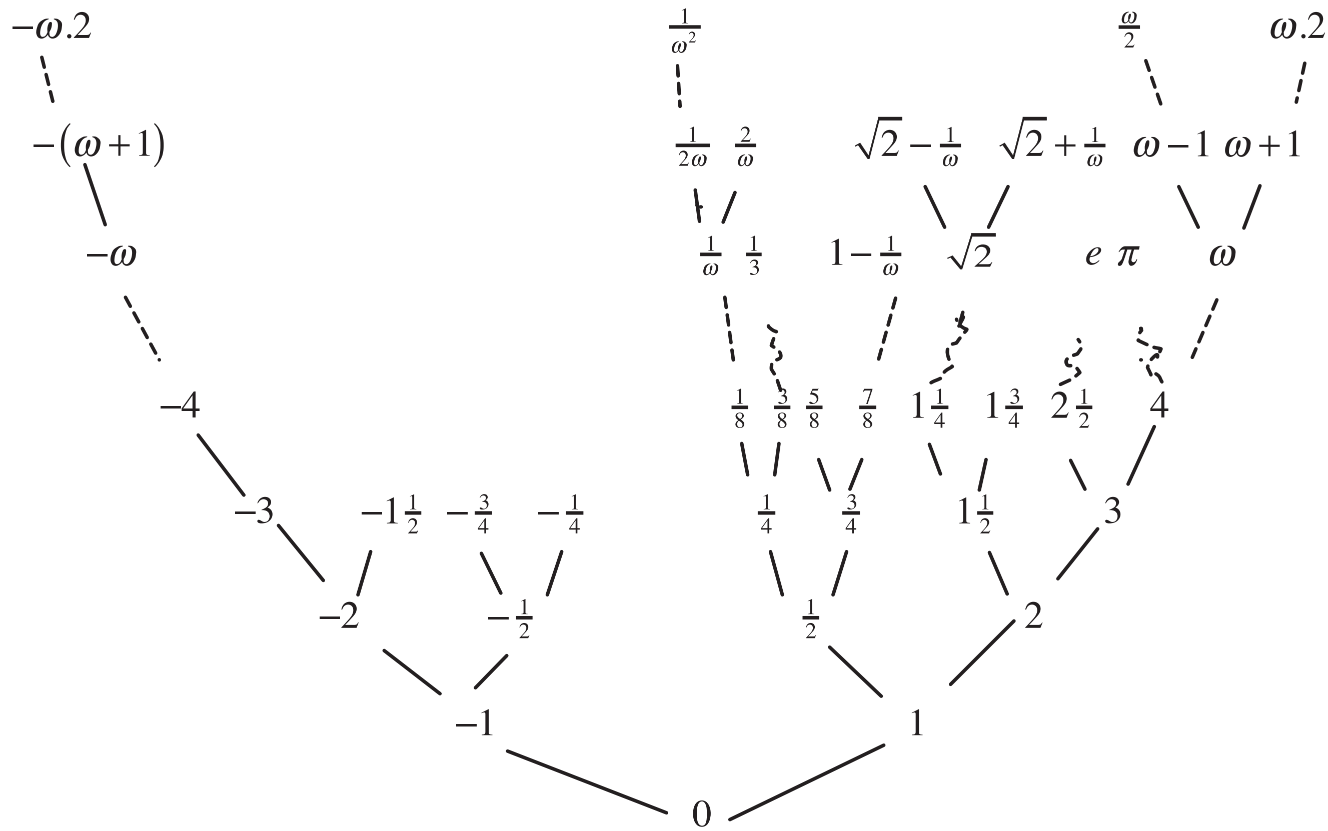


Figure 1. Some of the earliest created surreals.

and so on, each newly created surreal filling a cut in the ordered set of previously constructed surreal numbers. As is evident from the above, unlike Dedekind's cuts, the left and right sides of Conway's cuts may be empty. In fact, those surreal numbers having no right options, beginning with 0, 1, and 2, turn out to be **No**'s ordinals and those having no left options, beginning with $0, -1$ and -2, are the additive inverses of the ordinals, a surreal number and its additive inverse always being created on the same day. The surreal numbers other than the integers that emerge on finite days are the remainder of **No**'s dyadic rationals and those emerging on the ωth day are the non-dyadic real numbers as well as a host of other numbers including $-\omega = \{|..., -n, ..., -1, 0\}$, $\omega = \{0, 1, ..., n, ...|\}$, $-1/\omega = \{-1, -1/2, ... -1/2^n, ...|0\}$ and $1/\omega = \{0|..., 1/2^n, ..., 1/2, 1\}$, to name a few. For a slightly broader glimpse of some of the earliest created surreal numbers, see Figure 1, which is essentially taken from [5].

As the above description of Conway's construction suggests, the system of surreal numbers has a rich algebraico-tree-theoretic structure that emerges from combining Conway's field operations with **No**'s structure as a tree. It is the marriage of these two components, which we now consider in turn, from which the appellations for surreal numbers as well as a host of other distinctive features of **No** accrue.

Using the familiar definitions of = and < in terms of the partial ordering ≥ that is anteriorly defined on games, Conway shows that **No** is an ordered field when $+, -$, and $\cdot$ are defined by the following inductive stipulations, where x^L, y^L, x^R, and y^R are understood to range over the left and right options of x and y.

Definition of $x + y$.

$$x + y = \{x^L + y, x + y^L | x^R + y, x + y^R\}.$$

Definition of $-x$.

$$-x = \{-x^R | -x^L\}.$$

Definition of xy.

$$\begin{aligned} xy \;=\; & \{x^L y + x y^L - x^L y^L, x^R y + x y^R - x^R y^R | \\ & x^L y + x y^R - x^L y^R, x^R y + x y^L - x^R y^L\}. \end{aligned}$$

Conway shows that for each surreal $x = \{L|R\}$, x is the *earliest created* surreal number lying between the members of L and the members of R, i.e., x is the earliest constructed $z \in$ **No** such that $L < \{z\} < R$. Appealing to this result a significant portion of the theoretical underpinnings of the above definitions of + and $\cdot$ may be brought to the fore by the following variations on observations due to Conway.

Since $x = \{x^L | x^R\}$ and $y = \{y^L | y^R\}$, it follows that for all x^L, x^R, y^L and y^R:

$$(*) \quad x^L < x < x^R \text{ and } y^L < y < y^R.$$

Accordingly, for **No** to be an ordered additive group, it must be the case that

$$(**) \quad x^L + y, x + y^L < x + y < x^R + y, x + y^R,$$

for all x^L, x^R, y^L, and y^R. Therefore, since $x+y$ must lie between the two sets of inductively defined members of **No** specified in $(**)$ if **No** is to be an ordered group, the above definition of $x+y$ deems $x+y$ to be the earliest created surreal number consistent with that intended outcome. Similarly, for **No** to be an ordered field, it follows from $(*)$ that each of the differences $x - x^L, x^R - x, y - y^L, y^R - y$ must be positive and, hence, likewise each of the products $(x - x^L)(y - y^L)$, $(x^R - x)(y^R - y)$, $(x - x^L)(y^R - y)$ and $(x^R - x)(y - y^L)$. And so by applying the routine algebra of ordered fields to each of these products one obtains for all x^L, x^R, y^L and y^R:

$$(***) \quad x^L y + x y^L - x^L y^L, x^R y + x y^R - x^R y^R < \\ xy < x^L y + x y^R - x^L y^R, x^R y + x y^L - x^R y^L.$$

Consequently, since xy must lie between the sets of inductively defined members of **No** specified in $(***)$ if **No** is to be an ordered field, the above definition of xy requires xy to be the earliest constructed surreal compatible with that desired end.

While the above observations reveal the incisive nature of Conway's cryptic-seeming inductive definitions, there is no a priori reason to believe that these definitions, however cleverly motivated, would lead to the existence of an ordered field, let alone one having the rich structure of **No**. That they do is one of Conway's most remarkable surreal discoveries.

As we mentioned above, in addition to its inclusive structure as an ordered field, **No** has a rich tree-theoretic structure, a *tree* being a partially ordered class $(A, <_A)$ such that for each $x \in A$ the class $\{y \in A : y <_A x\}$ of *predecessors* of x is a well-ordered set. This *simplicity hierarchical* (or *s-hierarchical*) *structure*, as it is sometimes called, is introduced by Conway by associating each surreal with a unique sequence of $+$'s and $-$'s indexed over an ordinal, called it's *sign-expansion*, but can be introduced more concisely as follows using the canonical representation of surreal numbers employed in Conway's treatment.

Each surreal x has a unique representation $\{L_x | R_x\}$, where (L_x, R_x) is a pair of (possibly empty) collectively exhaustive subsets of the set of all surreal numbers constructed at earlier stages of the construction. Using this representation, the *surreal number tree* $(\mathbf{No}, <_s)$ is obtained by stipulating that for all $x, y \in \mathbf{No}$, $x <_s y$ (read "x is *simpler than* y) if and only if $L_x < \{y\} < R_x$ and $x \neq y$.[1] $(\mathbf{No}, <_s)$ is in fact a *full binary tree*, that is, every member of **No** has two immediate successors and every chain in $(\mathbf{No}, <_s)$ of limit length (including the empty chain) has a unique immediate successor. In particular, for each surreal number x, $\{L_x | \{x\} \cup R_x\}$ and $\{L_x \cup \{x\} | R_x\}$ are the immediate successors of x, and if $(x_\alpha)_{\alpha<\beta}$ is a chain in $(\mathbf{No}, <_s)$ indexed over a limit ordinal β, then $\{\bigcup_{\alpha<\beta} L_{x_\alpha} | \bigcup_{\alpha<\beta} R_{x_\alpha}\}$ is the immediate successor of the chain, $\{\emptyset|\emptyset\}$ being the immediate successor of the empty chain.

As we alluded to above, among the striking s-hierarchical features of **No** is that every surreal number can be assigned a canonical "proper name" that is a reflection of its characteristic s-hierarchical properties. These *Conway names*, or *normal forms* as Conway calls them, are expressed as formal sums of the form

$$\sum_{\alpha<\beta} r_\alpha \omega^{y_\alpha},$$

where β is an ordinal, $(y_\alpha)_{\alpha<\beta}$ is a strictly decreasing sequence of surreals, and $(r_\alpha)_{\alpha<\beta}$ is a sequence of nonzero real numbers. Every such formal sum is in fact the Conway name of a surreal number, the Conway name of an ordinal being just its *Cantor normal form*.

In light of the above, we see that Figure 1 in fact offers a glimpse of the some of the earliest created members of $(\mathbf{No}, <_s)$ expressed in terms of their Conway names, where, for example, ω is the least infinite ordinal as well as *the simplest positive infinite number*, $-\omega$ is the additive inverse of ω as well as *the simplest negative infinite number* and $1/\omega$ is the multiplicative inverse of ω as well as *the simplest positive infinitesimal number*. It is worth noting, that being a real-closed field, the Conway names for **No**'s real algebraic numbers are determined solely by **No**'s algebraic structure, whereas the Conway names for the remaining surreals are fixed by algebraico-tree-theoretic considerations (e.g., [8, pp. 1244–1248]).

Conway observed that when the surreals are expressed in terms of their normal forms, **No** assumes the structure

[1] *As we mentioned above, Conway shows that each surreal number $\{L|R\}$ is the earliest created surreal number lying between its left and right options, the latter being a special case of his result for games, where, however, one cannot in general say "a game lies between its left and right options." In* [5, p. 23], *Conway dubs the earliest created such game (surreal number), the* simplest *such game (surreal number), naturally suggesting that for surreal numbers, as for games, "x is simpler than y" should be interpreted as x is constructed prior to y. In 1985 Conway agreed with the author that the simpler than relation for the surreals should be defined in terms of the predecessor relation in the tree rather than in terms of the created earlier than relation. This understanding was brought to the fore in* [8] *and has since emerged as the dominant interpretation of the simpler than relation in research on surreal numbers. Whereas a surreal number $\{L|R\}$ continues to be both the simplest and the earliest created surreal number lying between its left and right options, the meaning of "simpler than" has changed. For example, whereas $1/2$ is created earlier than ω, $1/2$ is not simpler than ω in the tree-theoretic sense (see Figure 1).*

of an ordered field of generalized formal power series with sums and products defined like polynomials and order defined lexicographically. In addition to making the surreals more tractable from an algebraic point of view, this permitted Conway to apply to **No** insights about such structures that accrue from the classical works of Hans Hahn (1907) and B. H. Neumann (1949) on the number systems and generalizations thereof that emerged from non-Archimedean geometry, and thereby relate the surreals to one of the roots of non-Archimedean mathematics.

Making use of Conway names, Conway also characterized the notion of *integer* appropriate to **No**. The discrete ring **Oz** of *omnific integers*, which extends **On**, consists of the surreal numbers whose Conway names have nonnegative exponents and integer coefficients when the exponent is 0. Every surreal number is distant at most 1 from some omnific integer, and **No** is **Oz**'s field of fractions.

Another striking s-hierarchical feature of **No** is that, much as the surreal numbers emerge from the empty set of surreal numbers by means of a transfinite induction that generates the entire spectrum of "numbers great and small," the inductive process of defining **No**'s arithmetic in turn generates the entire spectrum of ordered fields (ordered abelian groups) in such a way that an isomorphic copy of every such system either emerges as an *initial substructure* of **No** — a substructure A in which the tree-theoretic predecessors in A of each of its elements coincide with its predecessors in **No** — or is contained in a theoretically distinguished instance of such a system that does. In particular, as Ehrlich [8] showed, *every real-closed ordered field (divisible ordered abelian group) is isomorphic to an initial subfield (subgroup) of* **No**.

Since every real-closed field is isomorphic to an initial subfield of **No**, the underlying ordered field of every *hyperreal number system* — the nonstandand models of analysis employed in *Robinsonian* or *nonstandard analysis* — is isomorphic to an initial subfield of **No**. In fact, as Ehrlich [9] observed, **No** is isomorphic to the underlying ordered field of the richest hyperreal number system in NBG. On the other hand,"**No** is really irrelevant to nonstandard analysis," as Conway [5, p. 44] noted; and, vice versa. After all, whereas the transfer property of hyperreal number systems, a property not possessed by **No**, is central to the development of nonstandard analysis, the s-hierarchical structure of **No**, which is absent from hyperreal number systems, is central to the theory of surreal numbers. Of course, this does not preclude that down the line there might be cross-fertilization between the two theories.

However, while surrealists have thus far shown little interest in applying surreal numbers to nonstandard analysis or in providing an infinitesimalist approach to classical analysis based on surreal numbers more generally, a number of surrealists beginning with Norton, Kruskal, and Conway have fostered the idea of extending analysis to the entire surreal domain.

Building on work of B. H. Neumann, Conway observed that there is a notion of convergence in **No** for power series of infinitesimal surreals that can be expressed using Conway names, and that the various analytic functions could be defined on bounded portions of **No** using such power series whenever they converge in the appropriate sense. On the other hand, Conway originally expressed doubt that "reasonable" global definitions of exponentiation, logarithm, sine, and cosine could be defined on **No** [5, First Edition, p. 43]. Through the collective efforts of Kruskal, Norton, Gonshor, van den Dries, Ehrlich, and Kaplan, however, this doubt has been put to rest. Van den Dries and Ehrlich (2001) showed that **No** together with the Kruskal-Gonshor exponential function exp defined thereon [11] has the same elementary properties of the ordered field of real numbers with real exponentiation, and Ehrlich and Kaplan [10] have further shown that **No** has canonical sine and cosine functions which in turn lead to a canonical exponential function on **No**'s *surcomplex* counterpart **No**$[i]$ that extends exp.

Additional rudiments of analysis on the surreals have also been developed by Alling, Fornasiero, Rubinstein-Salzedo and Swaminathan, and Costin, Ehrlich and Friedman. Costin and Ehrlich, in particular, have developed a theory of integration (and differentiation) that extends the range of analysis from the reals to the surreals for a large subclass of *resurgent functions* that arise in applied analysis. The resurgent functions, which generalize the analytic functions, were introduced by Écalle in the early 1980s in connection with work related to Hilbert's 16th problem. Unlike nonstandard analysis, which provides an infinitesimalist approach to integration on the extended reals ($\mathbb{R} \cup \{\pm\infty\}$), surreal integration deals with integrals whose bounds and values need not be extended reals at all. For example, in the surreal theory (setting $e^x = \exp x$) we have

$$\int_0^{\omega} e^x dx = e^{\omega} - 1 = \omega^{\omega} - 1.$$

This work makes contributions towards realizing some of the analytic goals expressed by Kruskal and Norton in their unsuccessful early attempts to establish a theory of surreal integration as described by Conway in the Epilogue of [5, Second Edition].

Elements of asymptotic differential algebra — the subject that aims at understanding the asymptotics of solutions to differential equations from an algebraic point of view — have also been developed for the surreals. An *ordered differential field* is an ordered field K together with a *derivation* on K, i.e., a map $\partial : K \to K$ such that $\partial(a+b) = \partial(a) + \partial(b)$ and $\partial(ab) = \partial(a)b + a\partial(b)$ for all $a, b \in K$. Berarducci and Mantova (2018) constructed a derivation ∂_{BM} on **No** which has proven to have a number of desirable features, including $(\mathbf{No}, \partial_{BM})$ being universal with respect to a broad class of distinguished ordered differential fields. In their ICM talk [3], Aschenbrenner, van den Dries, and van der Hoeven outline the program they (along with Berarducci, Mantova, Bagayoko, and Kaplan) are engaged in for developing an ambitious theory of asymptotic differential algebra for all of **No**, though one that would require a derivation on **No** having compositional properties not enjoyed by ∂_{BM}. Such a program, if successful, would provide the most dramatic advance towards interpreting growth rates as numbers since the pioneering work of Paul du Bois-Reymond, G. H. Hardy, and Felix Hausdorff on "orders of infinity" in the decades bracketing the turn of the 20th century.

Work on the rates of growth of real functions, non-Archimedean geometry and Cantor's theory of the infinite are the primary sources of late nineteenth- and early twentieth-century non-Archimedean number systems. As the above remarks suggest, with his creation of the surreal numbers, Conway constructed a remarkable and profoundly original canonical framework for unifying not only these number systems but the reals and the underlying ordered fields of the hyperreal number systems to boot. With this, Conway joined the likes of Cantor, Dedekind, Hahn, and Robinson as one the foremost creators of systems of numbers great and small the world has ever known.

Conway's Tiling Groups

Richard Kenyon, Jeffrey Lagarias, and James Propp

John Conway was fascinated by tilings of the plane, from periodic tilings (the Conway criterion, orbifold notation for wallpaper groups) to aperiodic tilings (Penrose tilings, pinwheel tilings). Some of this work was described by Doris Schattschneider in her article "John Conway, Tilings, and Me" in the Summer 2021 special issue of the Intelligencer devoted to Conway's legacy. Less well-known is John's work on applying combinatorial group theory, a favorite tool of his, to the study of tiling problems in finite subregions of the plane. Conway and Lagarias' 1990 article "Tiling with Polyominoes and Combinatorial Group Theory" [6] opened a new door to the study of such problems. Thurston's 1990 article "Conway's tiling groups" added a geometric viewpoint on such invariants, introducing height functions, which we define below.

A *polyomino* is a connected union of a finite set of squares in an infinite square grid. We say that a collection of simply-connected polyominoes $T_1, \ldots, T_r$ (called *prototiles*) tiles a region R if we can write R as a union of polyominoes with disjoint interiors, each of which is a translate of one of the T_i. Recreational mathematics abounds in problems of the form "Do *these* prototiles tile *this* region?"; the classic problem of the genre is the Mutilated Checkerboard Problem, in which the prototiles are a 1-by-2 and a 2-by-1 rectangle (called dominos) and the region to be tiled is an 8-by-8 square from which two opposite 1-by-1 corner squares have been removed. The classic solution comes from a *coloring argument*, exploiting an alternating black-white coloring of the board: each domino covers one white square and one black square, but the mutilated checkerboard has unequal numbers of black and white squares, so no tiling exists. This argument was presented in Solomon Golomb's 1954 paper "Checker boards and polyominoes" which introduced the term "polyomino."

A more complicated problem of this kind, due to de Bruijn, calls for tiling a 6-by-6 square (or more generally a $(4m+2)$-by-$(4n+2)$ rectangle) with 1-by-4 and 4-by-1 prototiles; as in the case of the mutilated checkerboard, a naive area argument fails to solve the problem but a more sophisticated coloring argument (which the reader is invited to find) shows that no tiling exists.

Prior to 1990, the main ways to prove that a given tiling problem was unsolvable were to give a coloring argument, to conduct brute force examination of all possibilities, or to employ ad hoc methods that varied from problem to problem. The hope that a general method for solving such problems could be found was dashed by the work of Robert Berger, who in his 1966 article "The undecidability of the domino problem" showed that infinite tileability problems could be undecidable. For finite tilings, Leonid Levin showed in the 1973 paper "Universal search problems" that the class of finite tileability problems is NP-complete. Conway and Lagarias presented a new framework for tackling tiling problems that, while necessarily subject to the limitations imposed by these hardness results, went beyond what coloring arguments could do.

The basic idea, due to Conway, is to interpret paths in the square grid starting from the origin as elements of the free group F_2 on two generators A, U, where A ("across") represents a step to the right, A^{-1} a step left, U ("up") a step up, and U^{-1} a step down (backtracks naturally cancel under this correspondence). The free group product corresponds to concatenation of paths, translating one path

Richard Kenyon is Erastus L. DeForest Professor of Mathematics at Yale University. His email address is `richard.kenyon@yale.edu`.

Jeffrey Lagarias is a professor of mathematics at the University of Michigan. His email address is `lagarias@umich.edu`.

James Propp is a professor of mathematics at the University of Massachusetts, Lowell. His email address is `jamespropp@gmail.com`.

to the end of the other. For simply-connected regions R, the counterclockwise boundary ∂R of R is described by a word w in F_2, unambiguous up to a choice of base point for the loop (or, equivalently, unique up to conjugation). Likewise, the boundary of each prototile T_i corresponds to a word u_i. The main insight is that if a simply-connected R can be tiled by simply-connected prototiles T_i, then the word w can be expressed as a product of conjugates of the words u_i; in other words, if R can be tiled, then w is trivial in the quotient group $G = F_2/N$, where N is the normal subgroup of F_2 generated by conjugates of the tiles. Following Thurston we call G the *Conway tiling group* associated to the given prototile set, and the associated necessary condition ("boundary criterion") for R to have a tiling is that the boundary word w for ∂R belong to N.

A simple example (see Figure 2) illustrates the boundary criterion.

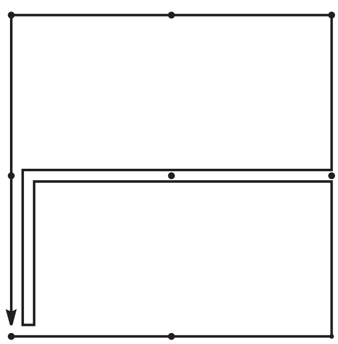


Figure 2. Composing boundary words, with conjugation.

Take R to be the 2-by-2 square $[0,2]\times[0,2]$, tiled by two 2-by-1 rectangles. We find that

$$w = A^2U^2A^{-2}U^{-2} = (A^2UA^{-2}U^{-1})\,U(A^2UA^{-2}U^{-1})U^{-1}$$

where the right side is the product of the boundary-words of the two constituent 2-by-1 rectangles, in which the second is conjugated by U.

It is not immediately clear that this algebraic criterion on tilings will be useful, since the word problem for groups is undecidable in general; however, there are many groups of geometric origin for which there are fast algorithms for decidability. The paper of Conway and Lagarias applied the group-theoretic condition to a tiling problem in the hexagonal lattice. Define a T_n-triangle as a polygon in a honeycomb grid composed of n rows of hexagons in which the ith row contains i hexagons ($1 \le i \le n$); for instance, Figure 3 shows T_n for the case $n = 9$.

The problem is to determine for which values of n the T_n-triangle can be tiled by copies of the T_2-triangle and the inverted T_2-triangle (as illustrated in Figure 3). The paper recast this as a problem in the square grid and then, using various algebraic and geometric arguments (including an invocation of the concept of winding number) showed

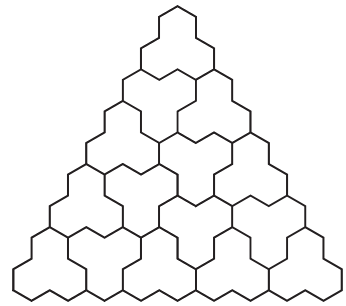


Figure 3. The T_9-triangle tiled by copies of the T_2-triangle and the inverted T_2-triangle.

that the values of n for which a tiling exists are the positive integers congruent to 0, 2, 9, or 11 (mod 12).

The paper also showed that *no possible coloring argument can prove this congruence criterion*. Coloring arguments that prove the nonexistence of a solution to a tiling problem will also prove nonexistence of a solution to the associated "signed tiling" problem. But the signed version of the triangle tiling problem has a different answer: it can be solved whenever n is congruent to 0 or 2 (mod 3).

The paper more generally described a connection between the coloring approach to tiling problems and the boundary invariants approach, observing that coloring arguments are covertly group-theoretic. They can be phrased in terms of quotient groups of the commutator subgroup $C = [F_2, F_2]$. The group C contains N as a normal subgroup, and C/N is analogous to a homotopy group. The abelianization of C/N is then analogous to a homology group, and coloring invariants are homology invariants.

Thurston's article "Conway's tiling groups" [17] takes a more geometric viewpoint, introducing the idea of associating to each tiling of R a function from the set of lattice-points inside R that lie on the boundaries of tiles to the Conway tiling group. In the case where the Conway tiling group is an extension of $\mathbb{Z}^2$ by $\mathbb{Z}$ (as holds for domino tilings) these functions are called *height functions*. Thurston uses height functions to obtain a necessary and sufficient condition for a (simply-connected) region R in the square lattice to have a domino tiling. The notion of height functions played a crucial role in the study of random tilings that exploded in the 1990s. The third author reported on these developments in the Conway memorial issue of the Mathematical Intelligencer (Summer 2021).

Skein Theory and More

Louis H. Kauffman

Here we discuss two key contributions of John Conway to knot theory: rational tangles for knots and links, and skein theory for knots and links. Conway's Tangle Theorem associates a tangle to each rational number; two of these will have the same topological type if and only if the rational numbers are equal. The knots we obtain by closing these tangles have the same topological type if the continued fraction expansions of the two rational numbers differ by a reversal of order.

In Figure 4, we illustrate some of the features of the theory of tangles. You will see tangles T and S, one labeled with the continued fraction $[2,3,4] = 2 + 1/(3 + 1/4) = 2 + 4/13 = 30/13$ and the other labeled with the continued fraction $[4,3,2] = 4 + 1/(3 + 1/2) = 4 + 2/7 = 30/7$. It turns out by Conway's Tangle Theorem [4] that these fractions classify the topological type of the tangles. For tangle-type we keep the ends fixed and let the tangles move about. Thus these two tangles are not topologically equivalent. But, as the figure shows, they are related. They both close to the same rational knot, labeled $N(T) = N(S)$ in the figure. Now notice that both of these fractions have the same numerator (30) and as for the denominators, we have that $7 \times 13 = 91$, a number that leaves a remainder of 1 on division by 30. These are not accidents. If $[a_1, a_2, \cdots, a_n]$ is the continued fraction for the tangle T and $[a_n, a_{n-1}, \cdots, a_1]$ (obtained by reversing the order of the terms) is the continued fraction for the tangle S, then both T and S close to form the same rational knot or link.

There is a beautiful way to classify rational knots (closures of rational tangles) from their continued fractions. We can take the symbol $(4,3,2)$, up to such reversal, as an indicator of the rational knot in the figure. (There is a futher technicality to the classification. See [14].) In this way Conway developed a simple notation to indicate rational knots and then used it along with insertion in certain graphs to make a very efficient notation for knots that lets us indicate thousands of knots in the knot tables with great elegance.

Here is a quick introduction to the skein theory of John Conway [4]. In Figure 5, I indicate one knot or link diagram K_+ and another diagram K_-, where it is understood that these two diagrams differ only by the switch that is illustrated for a single crossing. The same figure indicates another diagram K_0 where the crossing has been replaced by two parallel arcs. This is called a *smoothing* of

Louis H. Kauffman is a professor of mathematics emeritus at the University of Illinois at Chicago. His email address is `loukau@gmail.com`.

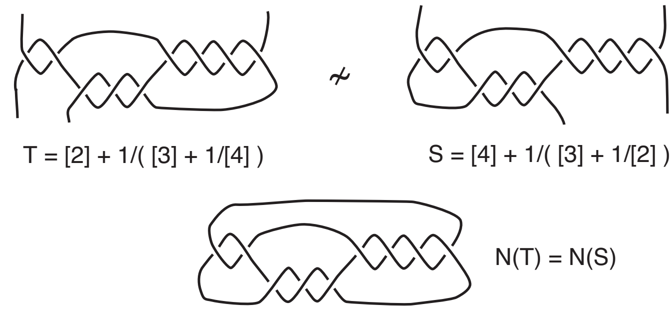


Figure 4. Continued Fractions and Rational Knots.

the crossing. The three diagrams K_+, K_-, K_0 taken together are called a *skein triple* and here is the key relationship for the *Alexander-Conway Polynomial* $\nabla_K(z)$ that is assigned to any oriented link diagram:

$$\nabla_{K_+} - \nabla_{K_-} = z\nabla_{K_0}.$$

Along with this *skein relation*, one has that

$$\nabla_O = 1$$

where O denotes an unknotted circle. If K and K' are topologically equivalent knots or links, then

$$\nabla_K = \nabla_{K'}.$$

This is the complete set of rules for finding the invariant $\nabla_K(z)$ for any oriented link K.

In Figure 6, we illustrate the simplest consequence of these axioms. The basic skein triple consists of U, U', V, where U and U' are unknots and V is a pair of unlinked circles. Each of U and U' evaluates to 1 and so their difference is 0. Thus we conclude that $z\nabla_V = 0$ and so $\nabla_V = 0$. In this way, an unlink V (of any number of components greater than 1) can be seen to receive the value 0 for its Alexander-Conway polynomial $\nabla_V(z)$.

In Figure 7, we indicate this situation. There T is a trefoil knot diagram and U is the result of switching one crossing in the diagram T. We can let $K_+ = T$, $K_- = U$ and $K_0 = L$, the link of two components illustrated in the top line of the figure. Thus we have

$$\nabla_T - \nabla_U = z\nabla_L$$

and

$$\nabla_L - \nabla_V = z\nabla_W.$$

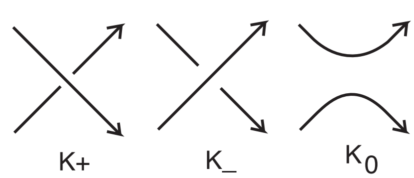


Figure 5. A SkeinTriple - three diagrams differing at one local site.

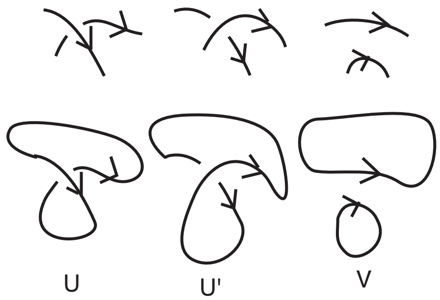


Figure 6. SkeinTriple for the UnLink.

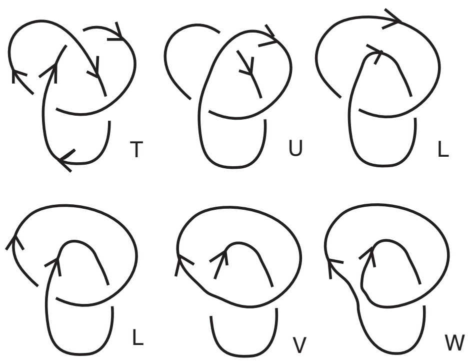


Figure 7. Trefoil Skein.

But we have that U is unknotted and V is unlinked, and W is unknotted. Furthermore we have checked that an unlink receives a zero polynomial. Thus we calculate that

$$\nabla_T - 1 = z\nabla_L$$

and

$$\nabla_L = z,$$

and so

$$\nabla_T = 1 + z^2.$$

Any knot or link diagram can be unknotted and unlinked by switching some of its crossings, just as we have done for the trefoil knot T and the Hopf link L. As a result, the Conway skein relation can be used to calculate the Alexander-Conway polynomial $\nabla_K(z)$ for any knot or link K. The remarkable fact is that, while there can be many different intermediate choices in this calculation, the answer is always unique and is a topological invariant of the link K.

One of the consequences of tangle theory and skein theory is that Conway was able to calculate invariants of knots and links very quickly. In Figure 8 we illustrate the well-known "Conway Knot" CK. This is an eleven crossing diagram of a non-trivial knot that has Alexander-Conway polynomial equal to 1. It is a smallest knot with this property. Is this knot the boundary of a disk embedded in four dimensional space? One says that a knot with this four dimensional property is *slice*. It was an open problem since

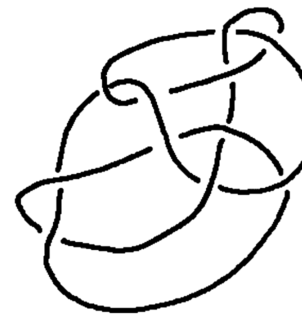


Figure 8. The Conway Knot of Alexander-Conway polynomial one.

Conway's work circa 1970 to determine whether the Conway knot is slice. This problem was recently resolved by Lisa Picarillo [16] in a stunning application of new invariants whose origins can be traced to Conway's original work. The knot is not slice.

Conway had a much more general notion of skein theory than the consequence of the one basic skein relation that we have quoted above. In this generalization, a knot or link K is placed in a "skein room" $\{K\}$ that represents its embedding in three dimensional space. Non-associative operations $\oplus, \ominus$ between skein rooms are defined so that

$$\{K_+\} = \{K_-\} \oplus \{K_0\}$$

and

$$\{K_-\} = \{K_+\} \ominus \{K_0\}.$$

Within a given room $\{K\}$, the knot or link K diagram can be deformed by ambient isotopy. When we examine a skein triple we take three representatives K_+, K_-, K_0 one from each of the rooms so that the representatives are exactly the same except in the places where they are switched or smoothed. We say that the $\{K_+\}$ room is *skein equivalent* to the concatenation $\{K_-\}\oplus\{K_0\}$. In this way, one can produce a *skein decompositon* of a knot or link. Refer to Figure 7 to see that we have

$$\{T\} = \{U\} \oplus \{L\},$$

$$\{L\} = \{V\} \oplus \{W\},$$

so that

$$\{T\} = \{U\} \oplus (\{V\} \oplus \{W\}).$$

This final skein decomposition of T expresses the trefoil knot in the skein as a composition of two unknots and an unlink. Every knot has such skein decomposition into unknots and unlinks. The non-associativity of the skein operations is crucial. Two knots or links are said to be *skein equivalent* if they have identical decompositions into unknots and unlinks. It is an open problem to this day to understand fully the skein equivalence classes of knots and links. In defining the skein, Conway opened a new area of topology.

In the context of the skein theory, the Alexander-Conway polynomial becomes a particular way to write down invariants of the skein so that

$$\nabla(A \oplus B) = \nabla(A) + z\nabla(B)$$

and

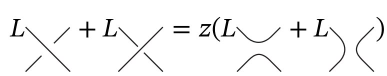
$$\nabla(A \ominus B) = \nabla(A) - z\nabla(B).$$

What was not obvious in the 1970s was the fact that there were other linear skein invariants than the Alexander-Conway polynomial and its multi-variable relatives. The most striking such invariant came in the wake of the Jones polynomial and is often called the Homflypt polynomial after its authors (in independent groups) Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki and Trawczk. The linear relation for Homflypt is

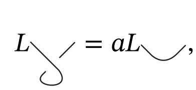
$$aP_{K_+} - a^{-1}P_{K_-} = zP_{K_0},$$

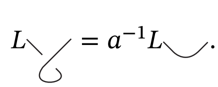
associating a Laurent polynomial $P_K(a,z)$ to an oriented knot or link K so that $P_K(a,z)$ is an invariant of the topological type of the knot. The Jones polynomial is a special case of the Homflypt polynomial. A key property of the Homflypt polynomial and the Jones polynomial is their ability to distinguish many knots from their mirror images. The background mathematical contexts that support these new skein polynomials involve many aspects of mathematical physics, Lie algebras and Hopf algebras. They are the background to more recent developments in Vassiliev invariants and link homology.

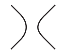
Certain aspects of skein theory came to light in relation to my own work. One was a model for the Alexander-Conway polynomial that used the work of Seifert from the 1930s [13]. Another is a state summation model for the Alexander-Conway Polynomial that is related to the original paper of J. W. Alexander [1, 12]. The state summation model is related to a state summation model (the Kauffman bracket state sum [13]) that I later discovered for the Jones polynomial. This state summation has a particularly simple form and a related unoriented skein expansion in the pattern shown below. The bracket can be seen as a special case of the so-called Kauffman two-variable polynomial denoted $L_K(a,z)$ with skein relation



and



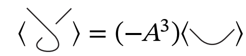


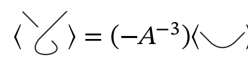
The bracket polynomial [13] model for the Jones polynomial can be described by an unoriented skein expansion of crossings  into *A-smoothings*  and *B-smoothings*  on a link diagram D via:

 (1)

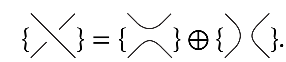
with

$$\langle D \bigcirc \rangle = (-A^2 - A^{-2})\langle D \rangle \quad (2)$$

 (3)

 (4)

In the sense of the Conway Skein Theory we have an unoriented skein with basic equation



By working with the equation at each crossing of a diagram for the knot, we obtain an unoriented skein decomposition where it is now understood that the operation $\oplus$ is neither commutative nor associative. Then the bracket polynomial becomes an evaluation on this unoriented skein satisfying the equation

$$\langle \{X\} \oplus \{Y\} \rangle = A\langle \{X\} \rangle + A^{-1}\langle \{Y\} \rangle.$$

Just as in the case of the oriented skein, this unoriented skein holds untapped mysteries that are slowly being revealed. It is a conjecture that the bracket polynomial detects the unknot, but it has been proved that the generalization of the bracket to a homology theory by Mikhail Khovanov does detect the unknot by work of Kronheimer and Mrowka. What else lies hidden in the oriented and the unoriented skeins for knots and links?

In Figure 9, we give a hint about the Khovanov homology. In that figure we illustrate all the states for the bracket state summation. This can also be construed as the full skein decomposition for the trefoil knot. Each diagram contributes a term to the bracket polynomial and the sum of these terms is the bracket polynomial. Khovanov examines this diagram of states and sees that it is a category. The objects of the category are the states themselves. The generating morphisms of the category are the arrows in the figure. Each arrow connects two states that differ by a smoothing at exactly one site, with the arrow going from the state with fewer B-smoothings to the state with one more B-smoothing. Khovanov defines his homology theory for knots by taking an appropriate homology theory for this category. Here we contact the roots of algebraic topology where the nerve of a category yields a simplicial structure and an appropriate functor from the category to a category of modules will send up rich possibilities of homological algebra. None of this would have come to pass if Conway had not found the skein.

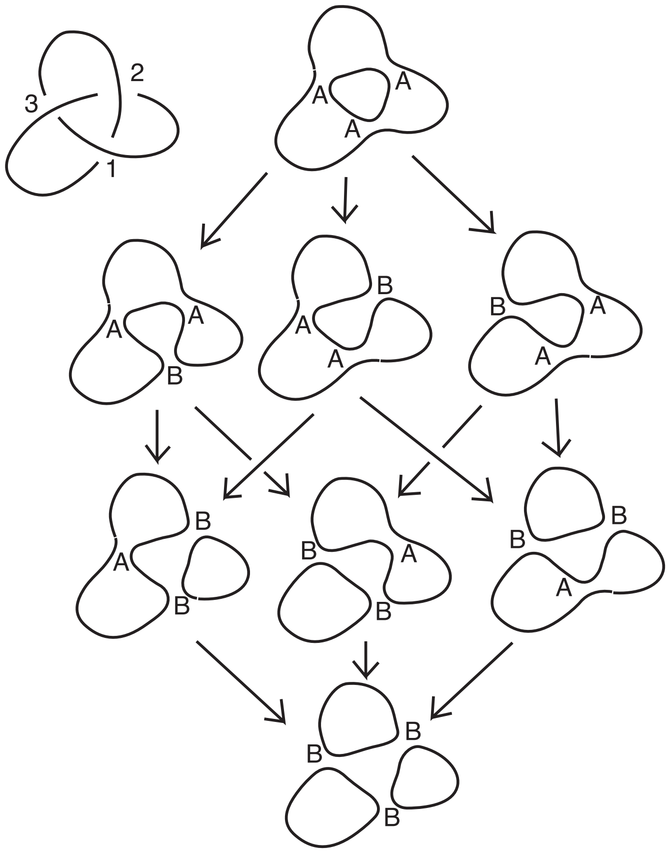


Figure 9. Bracket states and Khovanov Category - A category made from the states.

References

[1] J. W. Alexander, *Topological invariants of knots and links*, Trans. Amer. Math. Soc. **30** (1928), no. 2, 275–306, DOI 10.2307/1989123. MR1501429

[2] Norman L. Alling, *Foundations of analysis over surreal number fields*, North-Holland Mathematics Studies, vol. 141, North-Holland Publishing Co., Amsterdam, 1987. Notas de Matemática [Mathematical Notes], 117. MR886475

[3] Matthias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven, *On numbers, germs, and transseries*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 1–23. MR3966754

[4] J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), Pergamon, Oxford, 1970, pp. 329–358. MR0258014

[5] J. H. Conway, *On numbers and games*, London Mathematical Society Monographs, no. 6, Academic Press, London–New York, 1976; 2nd edition, A K Peters, Ltd., Natick, MA, 2001.

[6] J. H. Conway and J. C. Lagarias, *Tiling with polyominoes and combinatorial group theory*, J. Combin. Theory Ser. A **53** (1990), no. 2, 183–208, DOI 10.1016/0097-3165(90)90057-4. MR1041445

[7] Philip Ehrlich, *Universally extending arithmetic continua*, Le labyrinthe du continu (Cerisy-la-Salle, 1990), Springer, Paris, 1992, pp. 168–177. MR1413526

[8] Philip Ehrlich, *Number systems with simplicity hierarchies: a generalization of Conway's theory of surreal numbers*, J. Symbolic Logic **66** (2001), no. 3, 1231–1258, DOI 10.2307/2695104. MR1856739

[9] Philip Ehrlich, *The absolute arithmetic continuum and the unification of all numbers great and small*, Bull. Symbolic Logic **18** (2012), no. 1, 1–45. MR2798267

[10] Philip Ehrlich and Elliot Kaplan, *Surreal ordered exponential fields*, J. Symb. Log. **86** (2021), no. 3, 1066–1115, DOI 10.1017/jsl.2021.59. MR4347569

[11] Harry Gonshor, *An introduction to the theory of surreal numbers*, London Mathematical Society Lecture Note Series, vol. 110, Cambridge University Press, Cambridge, 1986, DOI 10.1017/CBO9780511629143. MR872856

[12] Louis H. Kauffman, *Formal knot theory*, Mathematical Notes, vol. 30, Princeton University Press, Princeton, NJ, 1983. MR712133

[13] Louis H. Kauffman, *New invariants in the theory of knots*, Amer. Math. Monthly **95** (1988), no. 3, 195–242, DOI 10.2307/2323625. MR935433

[14] L. H. Kauffman and S. Lambropoulou, *From tangle fractions to DNA*, in "Topology in Molecular Biology," Proceedings of the International Workshop and Seminar on Topology in Condensed Matter Physics, Dresden, 16–23 June 2002, edited by M. Monastyrsky, pp. 69–108.

[15] John McKay and Abdellah Sebbar, *Replicable functions: an introduction*, Frontiers in number theory, physics, and geometry. II, Springer, Berlin, 2007, pp. 373–386, DOI 10.1007/978-3-540-30308-4_10. MR2290767

[16] Lisa Piccirillo, *The Conway knot is not slice*, Ann. of Math. (2) **191** (2020), no. 2, 581–591, DOI 10.4007/annals.2020.191.2.5. MR4076631

[17] William P. Thurston, *Conway's tiling groups*, Amer. Math. Monthly **97** (1990), no. 8, 757–773, DOI 10.2307/2324578. MR1072815

Credits

Opening image is courtesy of CRC Press/Taylor & Francis Group.

Figure 1 is courtesy of Philip Ehrlich.

Figures 2 and 3 are courtesy of Richard Kenyon, Jeffrey Lagarias, and James Propp.

Figures 4–9 are courtesy of Louis H. Kauffman.