The Weingarten Calculus

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1. Introduction

Every compact topological group supports a unique translation invariant probability measure on its Borel sets — the *Haar measure*. The Haar measure was first constructed for certain families of compact matrix groups by Hurwitz in the nineteenth century in order to produce invariants of these groups by averaging their actions. Hurwitz's construction has been reviewed from a modern perspective by Diaconis and Forrester, who argue that it should be regarded as the starting point of modern random matrix theory [DF17]. An axiomatic construction of Haar measures in the more general context of locally compact groups was published by Haar in the 1930s, with further important contributions made in work of von Neumann, Weil, and Cartan; see [Bou04]. The existence of recent works on the

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Haar measure, see, e.g., [DS14] or [Mec19], can be seen as a token of the timeliness of this object as a modern research topic.

Given a measure, one wants to integrate. The Bochner integral for continuous functions F on a compact group G taking values in a given Banach space is called the *Haar integral*; it is almost always written simply

$$\int_G F(g)\,\mathrm{d}g,$$

with no explicit notation for the Haar measure. While integration on groups is a concept of fundamental importance in many parts of mathematics, including functional analysis and representation theory, probability, and ergodic theory, etc., the actual computation of Haar integrals is a problem which has received curiously little attention. As far as the authors are aware, it was first considered by theoretical physicists in the 1970s in the context of nonabelian gauge theories, where the issue of evaluating — or at least approximating — Haar integrals plays a major role. In particular, the physics literature on quantum chromodynamics, the main theory of strong interactions in particle physics, is littered with so-called "link integrals," which are Haar

integrals of the form

$$\int_{U(N)} U_{i(1)j(1)} \dots U_{i(d)j(d)} \overline{U_{i'(1)j'(1)} \dots U_{i'(d)j'(d)}} dU,$$

where $U(N)$ is the compact group of unitary matrices $U = [U_{xy}]_{x,y=1}^N$. Confronted with a paucity of existing mathematical tools for the evaluation of such integrals, physicists developed their own methods, which allowed them to obtain beautiful explicit formulas such as

$$\int_{U(N)} U_{11} U_{22} U_{33} \overline{U_{12} U_{23} U_{31}} dU = \frac{2}{N(N^2-1)(N^2-4)},$$

an evaluation which holds for all unitary groups of rank $N \geq 3$. Although exceedingly clever, the bag of tricks for evaluating Haar integrals assembled by physicists is ad hoc and piecemeal, lacking the unity and coherence which are the hallmarks of a mathematical theory.

The missing theory of Haar integrals began to take shape in the early 2000s, driven by an explosion of interest in random matrix theory. The basic Hilbert spaces of random matrix theory are $L^2(H(N), \text{Gauss})$ and $L^2(U(N), \text{Haar})$, where $H(N)$ is the noncompact abelian group of Hermitian matrices $H = [H_{xy}]_{x,y=1}^N$ equipped with a Gaussian measure of mean $\mu = 0$ and variance $\sigma > 0$, and $U(N)$ is the compact nonabelian group of unitary matrices $U = [U_{xy}]_{x,y=1}^N$ equipped with the Haar measure, just as above. Given a probability measure on some set of matrices, the basic goal of random matrix theory is to understand the induced distribution of eigenvalues, which in the selfadjoint case form a random point process on the line, and in the unitary case constitute a random point process on the circle. The *moment method* in random matrix theory, pioneered by Wigner ([Wig58]) in the 1950s, is an algebraic approach to this problem. The main idea is to adopt the algebra $\mathcal{S}$ of symmetric polynomials in eigenvalues as a basic class of test functions, and integrate such functions by realizing them as elements of the algebra $\mathcal{A}$ of polynomials in matrix elements, which can then (hopefully) be integrated by leveraging the defining features of the matrix model under consideration. The canonical example is sums of powers of eigenvalues (elements of $\mathcal{S}$), which may be represented as traces of matrix powers (elements of $\mathcal{A}$); more generally, all coefficients of the characteristic polynomial are sums of principal matrix minors.

It is straightforward to see that, in both of the above L^2-spaces, the algebra $\mathcal{A}$ of polynomial functions in matrix elements admits the orthogonal decomposition

$$\mathcal{A} = \bigoplus_{d=0}^{\infty} \mathcal{A}^{[d]}, \tag{1}$$

where $\mathcal{A}^{[d]}$ is the space of homogeneous degree d polynomial functions in matrix elements. Thus, modulo the algebraic issues inherent in transitioning from $\mathcal{S}$ to $\mathcal{A}$, linearity

of expectation reduces implementation of the method to computing scalar products of monomials of equal degree, which are expressions of the form

$$\left\langle \prod_{x=1}^d H_{i(x)j(x)}, \prod_{x=1}^d H_{i'(x)j'(x)} \right\rangle_{L^2(H(N), \text{Gauss})}$$

and

$$\left\langle \prod_{x=1}^d U_{i(x)j(x)}, \prod_{x=1}^d U_{i'(x)j'(x)} \right\rangle_{L^2(U(N), \text{Haar})}.$$

In the Gaussian case, monomial scalar products can be computed systematically using a combinatorial algorithm which physicists call the "Wick formula" and statisticians call the "Isserlis theorem." This device leverages independence together with the characteristic feature of centered normal distributions — vanishing of all cumulants but the second — to compute Gaussian expectations as polynomials in the variance parameter σ. The upshot is that scalar products in $L^2(H(N), \text{Gauss})$ are closely related to the combinatorics of graphs drawn on compact Riemann surfaces, which play the role of Feynman diagrams for selfadjoint matrix-valued field theories. We recommend ([Zvo97]) as an entry point into the fascinating combinatorics of Wick calculus.

The case of Haar unitary matrices is a priori more complicated: the random variables $\{U_{xy} : x, y \in [N]\}$ are identically distributed, thanks to the invariance of Haar measure, but they are also highly correlated, due to the constraint $U^*U = I$. Moreover, each individual entry follows a complicated law not uniquely determined by its mean and variance. Despite these obstacles, it turns out that, when packaged correctly, the invariance of Haar measure provides everything needed to develop an analogue of Wick calculus for Haar unitary matrices. Moreover, once the correct general perspective has been found, one realizes that it applies equally well to any compact group, and even to compact symmetric spaces and compact quantum groups. The resulting analogue of Wick calculus has come to be known as *Weingarten calculus*, a name chosen by Collins [Col03] to honor the contributions of Donald Weingarten, a physicist whose early work in the subject is of foundational importance.

The Weingarten calculus has matured rapidly over the course of the past decade, and the time now seems right to give a pedagogical account of the subject. The authors are currently preparing a monograph intended to meet this need. In this article, we aim to provide an easily digestible and hopefully compelling preview of our forthcoming work, emphasizing the big picture but still providing some of the important details.

First and foremost, we wish to impart the insight that, like the calculus of Newton and Leibniz, the core of

Weingarten calculus is a fundamental theorem which converts a computational problem into a symbolic problem: whereas the usual fundamental theorem of calculus converts the problem of integrating functions on the line into computing antiderivatives, the fundamental theorem of Weingarten calculus converts the problem of integrating functions on groups into computing certain matrices associated to tensor invariants. The fundamental theorem of Weingarten calculus is presented in detail in Section 2.

We then turn to examples illustrating the fundamental theorem in action. We present two detailed case studies: integration on the automorphism group S(N) of a finite set of size N, and integration on the automorphism group U(N) of N-dimensional Hilbert space. These are natural examples, given that the symmetric group and the unitary group are model examples of a finite and infinite compact group, respectively. The S(N) case, presented in Section 3, is a toy example chosen to illustrate how Weingarten calculus works in an elementary situation where the integrals to which it applies can easily be evaluated from first principles. The U(N) case, discussed in Section 4, is an example of real interest, and we give a detailed workup showing how Weingarten calculus handles the link integrals of U(N) lattice gauge theory.

Section 5 gives a necessarily brief discussion of Weingarten calculus for the remaining classical groups, namely the orthogonal group O(N) and the symplectic group Sp(N), both of which receive a detailed treatment in a book in preparation by the authors. Finally, Section 6 extols the universality of Weingarten calculus, briefly discussing how it can be transported to compact symmetric spaces and compact quantum groups, and indicating applications in quantum information theory.

2. The Fundamental Theorem

Given a compact group G, a finite-dimensional Hilbert space $\mathcal{H}$ with a specified orthonormal basis $e_1, \dots, e_N$, and a continuous group homomorphism $U : \text{G} \to \text{U}(\mathcal{H})$ from G to the unitary group of $\mathcal{H}$, let $U_{xy} : \text{G} \to \mathbb{C}$ be the corresponding matrix element functionals,

$$U_{xy}(g) = \langle e_x, U(g)e_y \rangle, \quad 1 \leq x, y \leq N.$$

The *Weingarten integrals* of the unitary representation $(\mathcal{H}, U)$ are the integrals

$$I_{ij} = \int_{\text{G}} \prod_{x=1}^{d} U_{i(x)j(x)}(g)\text{d}g,$$

where d ranges over the set $\mathbb{N}$ of positive integers, and the multi-indices i, j range over the set $\text{Fun}(d, N)$ of functions from $[d] = \{1, \dots, d\}$ to $[N] = \{1, \dots, N\}$. Clearly, if we can compute all Weingarten integrals I_{ij}, then we can integrate any function on G which is a polynomial in the matrix elements U_{xy}. This is the basic problem of Weingarten

calculus: compute the Weingarten integrals of a given unitary representation of a given compact group.

The fundamental theorem of Weingarten calculus addresses this problem by linearizing it. The basic observation is that, for each $d \in \mathbb{N}$, the N^{2d} integrals I_{ij}, $i, j \in \text{Fun}(d, N)$, are themselves the matrix elements of a linear operator. Indeed, we have

$$I_{ij} = \int_{\text{G}} U_{ij}^{\otimes d}(g)\text{d}g,$$

where

$$e_i = e_{i(1)} \otimes \cdots \otimes e_{i(d)}, \quad i \in \text{Fun}(d, N) \tag{2}$$

is the orthonormal basis of $\mathcal{H}^{\otimes d}$ corresponding to the specified orthonormal basis $e_1, \dots, e_N$ in $\mathcal{H}$, and

$$U_{ij}^{\otimes d}(g) = \langle e_i, U^{\otimes d}(g)e_j \rangle, \quad i, j \in \text{Fun}(d, N),$$

are the matrix elements of the unitary operator $U^{\otimes d}(g)$ in this basis. We thus have that

$$I_{ij} = P_{ij}, \quad i, j \in \text{Fun}(d, N),$$

where $P_{ij} = \langle e_i, Pe_j \rangle$ are the matrix elements of the selfadjoint operator

$$P = \int_{\text{G}} U^{\otimes d}(g)\text{d}g$$

obtained by integrating the unitary operators $U^{\otimes d}(g)$ against Haar measure. The basic problem of Weingarten calculus is thus equivalent to computing the matrix elements of $P \in \text{End} \, \mathcal{H}^{\otimes d}$, for all $d \in \mathbb{N}$.

This is where the characteristic feature of Haar measure, the invariance

$$\int_{\text{G}} F(g_0 g)\text{d}g = \int_{\text{G}} F(gg_0)\text{d}g = \int_{\text{G}} F(g)\text{d}g, \quad g_0 \in \text{G},$$

comes into play: it forces $P^2 = P$. Thus P is a selfadjoint idempotent, and as such P orthogonally projects $\mathcal{H}^{\otimes d}$ onto its image, which is the space of G-invariant tensors in $\mathcal{H}^{\otimes d}$,

$$(\mathcal{H}^{\otimes d})^{\text{G}} = \{t \in \mathcal{H}^{\otimes d} : U^{\otimes d}(g)t = t \text{ for all } g \in \text{G}\}.$$

Thus, we see that the basic problem of Weingarten calculus is in fact very closely related to the basic problem of invariant theory, which is to determine a basis for the space of G-invariant tensors in $\mathcal{H}^{\otimes d}$ for all $d \in \mathbb{N}$.

Indeed, suppose we have access to a basis $a_1, \dots, a_m$ of $(\mathcal{H}^{\otimes d})^{\text{G}}$. Then, by elementary linear algebra, we have everything we need to calculate the matrix

$$\mathbf{P} = [I_{ij}]_{i,j \in \text{Fun}(d, N)}$$

of degree d Weingarten integrals. Let $\mathbf{A}$ be the $N^d \times m$ matrix whose columns are the coordinates of the basic invariants in the desired basis,

$$\mathbf{A} = [\langle e_i, a_x \rangle]_{i \in \text{Fun}(d, N), x \in [m]}.$$

Then we have the matrix factorization

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*,$$

familiar from matrix analysis as the multidimensional generalization of the undergraduate "outer product divided by inner product" formula for orthogonal projection onto a line. The $m \times m$ matrix $\mathbf{A}^*\mathbf{A}$ is nothing but the Gram matrix

$$\mathbf{A}^*\mathbf{A} = [\langle a_x, a_y \rangle]_{x,y \in [m]}$$

of the basic G-invariants in $\mathcal{H}^{\otimes d}$, whose linear independence is equivalent to the invertibility of the Gram matrix. Let us give the inverse Gram matrix a name: we call

$$\mathbf{W} = (\mathbf{A}^*\mathbf{A})^{-1}$$

the *Weingarten matrix* of the invariants $a_1, \dots, a_m$. Extracting matrix elements on either side of the factorization $\mathbf{P} = \mathbf{A}\mathbf{W}\mathbf{A}^*$, we obtain the Fundamental Theorem of Weingarten Calculus.

Theorem 2.1. *For any $d \in \mathbb{N}$ and $i, j \in \mathrm{Fun}(d, N)$, we have*

$$I_{ij} = \sum_{x,y=1}^{m} \mathbf{A}_{ix} \mathbf{W}_{xy} \mathbf{A}_{yj}^*.$$

Does Theorem 2.1 actually solve the basic problem of Weingarten calculus? Yes, insofar as the classical fundamental theorem of calculus solves the problem of computing definite integrals: it reduces a numerical problem to a symbolic problem. In order to apply the fundamental theorem of calculus to integrate a given function, one must find its antiderivative, and as every student of calculus knows this can be a wild ride. In order to use the fundamental theorem of Weingarten calculus to compute the Weingarten integrals of a given unitary representation, one must solve a souped-up version of the basic problem of invariant theory which involves not only finding basic tensor invariants, but computing their Weingarten matrices. Just like the computation of antiderivatives, this may prove to be a difficult task.

3. The Symmetric Group

In this Section, we consider a toy example. Fix $N \in \mathbb{N}$, and let $\mathrm{S}(N)$ be the symmetric group of rank N, viewed as the group of bijections $g : [N] \to [N]$. This is a finite group, its topology and resulting Haar measure are discrete, and all Haar integrals are finite sums. We will solve the basic problem of Weingarten calculus for the permutation representation of $\mathrm{S}(N)$ in two ways: using elementary combinatorial reasoning, and using the fundamental theorem of Weingarten calculus. It is both instructive and psychologically reassuring to work through the two approaches and see that they agree.

The permutation representation of $\mathrm{S}(N)$ is the unitary representation $(\mathcal{H}, U)$ in which $\mathcal{H}$ is an N-dimensional

Hilbert space with orthonormal basis $e_1, \dots, e_N$, and $U : \mathrm{S}(N) \to \mathrm{U}(\mathcal{H})$ is defined by

$$U(g)e_x = e_{g(x)}, \quad x \in [N].$$

The corresponding system of matrix elements $U_{xy} : \mathrm{S}(N) \to \mathbb{C}$ is given by

$$U_{xy}(g) = \langle e_x, U(g)e_y \rangle = \delta_{xg(y)}, \quad x, y \in [N].$$

We will evaluate the Weingarten integrals of $(\mathcal{H}, U)$,

$$I_{ij} = \int_{\mathrm{S}(N)} \prod_{x=1}^{d} U_{i(x)j(x)}(g)\mathrm{d}g.$$

Each Weingarten integral I_{ij} is a finite sum with $N!$ terms, each equal to zero or one:

$$I_{ij} = \frac{1}{N!} \sum_{g \in \mathrm{S}(N)} \prod_{x=1}^{d} U_{i(x)j(x)}(g)$$

$$= \frac{1}{N!} \sum_{g \in \mathrm{S}(N)} \prod_{x=1}^{d} \delta_{g^{-1}i(x),j(x)}.$$

Thus, $N! \, I_{ij}$ simply counts permutations $g \in \mathrm{S}(N)$ which solve the equation $g^{-1}i = j$. This is an elementary counting problem, and a good way to solve it is to think of the given functions $i, j \in \mathrm{Fun}(d, N)$ "backwards," as the ordered lists of their fibers:

$$i = (i^{-1}(1), \dots, i^{-1}(N))$$
$$j = (j^{-1}(1), \dots, j^{-1}(N)).$$

The fiber fingerprint of the composite function $g^{-1}i \in \mathrm{Fun}(d, N)$ is then

$$g^{-1}i = (i^{-1}(g(1)), \dots, i^{-1}(g(N))),$$

and so we have $g^{-1}i = j$ if and only if

$$(i^{-1}(g(1)), \dots, i^{-1}(g(N))) = (j^{-1}(1), \dots, j^{-1}(N)).$$

Clearly, such a permutation exists if and only if the fibers of i and j are the same up to the labels of their base points, which is the case if and only if

$$\mathrm{type}(i) = \mathrm{type}(j),$$

where $\mathrm{type}(i)$ is the partition of $[d]$ obtained by forgetting the order on the fibers of i and throwing away empty fibers. The permutations we wish to count thus number

$$\delta_{\mathrm{type}(i)\mathrm{type}(j)}(N - \#\mathrm{type}(i))! \tag{3}$$

in total, where $\#\pi$ denotes the number of blocks of the set partition π. We conclude that the integral I_{ij} is given by

$$I_{ij} = \delta_{\mathrm{type}(i)\mathrm{type}(j)} \frac{(N - \#\mathrm{type}(i))!}{N!}$$

$$= \frac{\delta_{\mathrm{type}(i)\mathrm{type}(j)}}{N(N-1)\dots(N - \#\mathrm{type}(i) + 1)}. \tag{4}$$

Let us now evaluate I_{ij} using the Fundamental Theorem of Weingarten Calculus. The first step is to solve the basic

problem of invariant theory for the representation $(\mathcal{H}, U)$. This is again straightforward. Fix $d \in \mathbb{N}$, let $\mathsf{Par}_N(d)$ denote the set of partitions of $[d]$ with at most N blocks, and to each $\mathsf{p} \in \mathsf{Par}_N(d)$ associate the tensor

$$a_\mathsf{p} = \sum_{\substack{i \in \mathrm{Fun}(d,N) \\ \mathrm{type}(i)=\mathsf{p}}} e_i,$$

where $e_i = e_{i(1)} \otimes \cdots \otimes e_{i(d)} \in \mathcal{H}^{\otimes d}$. It is apparent that the set $\{a_\mathsf{p} : \mathsf{p} \in \mathsf{Par}_N(d)\}$ is a basis of $(\mathcal{H}^{\otimes d})^{\mathrm{S}(N)}$. Indeed, taking the unit tensor

$$e_i = e_{i(1)} \otimes \cdots \otimes e_{i(d)}$$

corresponding to a function $i \in \mathrm{Fun}(d, N)$ and symmetrizing it using the action of permutations on multi-indices produces the tensor

$$a_i = \sum_{g \in \mathrm{S}(N)} e_{gi(1)} \otimes \cdots \otimes e_{gi(d)},$$

which is clearly $\mathrm{S}(N)$-invariant, and moreover it is clear that every $\mathrm{S}(N)$-invariant tensor in $\mathcal{H}^{\otimes d}$ is a linear combination of tensors of this form. Furthermore,

$$a_i = a_j \iff \mathrm{type}(i) = \mathrm{type}(j),$$

so that the distinct invariants produced by symmetrization of the initial basis in $\mathcal{H}^{\otimes d}$ are

$$a_\mathsf{p} = \sum_{\substack{i \in \mathrm{Fun}(d,N) \\ \mathrm{type}(i)=\mathsf{p}}} e_i, \quad \mathsf{p} \in \mathsf{Par}_N(d).$$

These tensors are pairwise orthogonal: for any $\mathsf{p}, \mathsf{q} \in \mathsf{Par}_N(d)$, we have

$$\langle a_\mathsf{p}, a_\mathsf{q} \rangle = \left\langle \sum_{i \in \mathrm{Fun}(d,N)} \delta_{\mathrm{type}(i)\mathsf{p}} e_i, \sum_{j \in \mathrm{Fun}(d,N)} \delta_{\mathrm{type}(j)\mathsf{q}} e_j \right\rangle$$

$$= \sum_{i \in \mathrm{Fun}(d,N)} \sum_{j \in \mathrm{Fun}(d,N)} \delta_{\mathrm{type}(i)\mathsf{p}} \delta_{\mathrm{type}(j)\mathsf{q}} \delta_{ij}$$

$$= \delta_{\mathsf{p}\mathsf{q}} \sum_{i \in \mathrm{Fun}(d,N)} \delta_{\mathrm{type}(i)\mathsf{p}}$$

$$= \delta_{\mathsf{p}\mathsf{q}} N(N-1)\dots(N-\#(\mathsf{p})+1).$$

So, the Gram matrix of the basis $\{a_\mathsf{p} \in \mathsf{Par}_N(d)\}$ is diagonal, and the corresponding Weingarten matrix $\mathbf{W}$ has entries

$$\mathbf{W}_{\mathsf{p}\mathsf{q}} = \frac{\delta_{\mathsf{p}\mathsf{q}}}{N(N-1)\dots(N-\#(\mathsf{p})+1)}.$$

We can now apply the fundamental Theorem of Weingarten calculus to obtain

$$I_{ij} = \int_{\mathrm{S}(N)} U_{i(1)j(1)} \dots U_{i(d)j(d)} \mathrm{d}g$$

$$= \sum_{\mathsf{p},\mathsf{q} \in \mathsf{Par}_N(d)} \langle e_i, a_\mathsf{p} \rangle \mathbf{W}_{\mathsf{p}\mathsf{q}} \langle a_\mathsf{q}, e_j \rangle$$

$$= \sum_{\mathsf{p},\mathsf{q} \in \mathsf{Par}_N(d)} \frac{\delta_{\mathrm{type}(i),\mathsf{p}} \delta_{\mathsf{p}\mathsf{q}} \delta_{\mathsf{q},\mathrm{type}(j)}}{N(N-1)\dots(N-\#(\mathfrak{p})+1)}$$

$$= \frac{\delta_{\mathrm{type}(i)\mathrm{type}(j)}}{N(N-1)\dots(N-\#\mathrm{type}(i)+1)}.$$

4. The Unitary Group

In this section we consider a case of real interest: integration on the unitary group $\mathrm{U}(N)$ of an N-dimensional Hilbert space $\mathcal{H}$. The most obvious unitary representation of this group is the tautological representation $(\mathcal{H}, U)$, in which $U(g) = g$. Relative to an orthonormal basis $e_1, \dots, e_N$ of $\mathcal{H}$, the resulting system of matrix elements $U_{xy} : \mathrm{G} \to \mathbb{C}$ is simply

$$U_{xy}(g) = \langle e_x, g e_y \rangle, \quad 1 \le x, y \le N,$$

and it turns out that all corresponding Weingarten integrals

$$I_{ij} = \int_{\mathrm{U}(N)} \prod_{x=1}^d U_{i(x)j(x)}(g) \mathrm{d}g$$

vanish. To see this, let λ_0 be an arbitrary complex number of modulus one, and let $g_0 \in \mathrm{U}(N)$ be the scalar operator with eigenvalue λ_0. We then have $U_{xy}(gg_0) = \lambda_0 U_{xy}(g)$, so invariance of Haar measure implies $I_{ij} = \lambda_0^d I_{ij}$, which forces $I_{ij} = 0$.

The basic problem of Weingarten calculus becomes much more interesting when when we replace the tautological representation with the adjoint representation. The carrier space of the adjoint representation is the algebra $\mathrm{End}\,\mathcal{H}$ of all linear maps $A : \mathcal{H} \to \mathcal{H}$ equipped with the Hilbert-Schmidt scalar product

$$\langle A, B \rangle = \mathrm{Tr}\, A^* B,$$

and the action V of $\mathrm{U}(N)$ on this Hilbert space is conjugation,

$$V(g)A = gAg^{-1}.$$

The orthonormal basis $e_1, \dots, e_N$ in $\mathcal{H}$ induces an orthonormal basis in $\mathrm{End}\,\mathcal{H}$ consisting of the N^2 matrix units defined by

$$E_{xx'} e_z = e_x \langle e_{x'}, e_z \rangle, \quad x, x', z \in [N].$$

The matrix units relate the scalar product on $\mathrm{End}\,\mathcal{H}$ to that on $\mathcal{H}$ via

$$\langle E_{xx'}, A \rangle = \langle e_x, A e_{x'} \rangle.$$

The matrix elements of the adjoint representation are thus related to those of the tautological representation by

$$
\begin{aligned}
V_{yy'xx'}(g) &= \langle E_{yy'}, V(g)E_{xx'} \rangle \\
&= \langle g^{-1}e_y, E_{xx'}g^{-1}e_{y'} \rangle \\
&= \sum_z \langle g^{-1}e_y, E_{xx'}e_z \rangle \langle e_z, g^{-1}e_{y'} \rangle \\
&= \sum_z \langle g^{-1}e_y, e_x \rangle \langle e_{x'}, e_z \rangle \langle e_z, g^{-1}e_{y'} \rangle \\
&= U_{xy}(g)U_{x'y'}(g^{-1}).
\end{aligned}
$$

So, the Weingarten integrals

$$
I_{jj'ii'} = \int_{U(N)} \prod_{x=1}^{d} V_{j(x)j'(x)i(x)i'(x)}(g)\mathrm{d}g,
$$

of the adjoint representation of $U(N)$ are exactly the link integrals

$$
L_{ii'jj'} = \int_{U(N)} \prod_{x=1}^{d} U_{i(x)i'(x)}(g)U_{j(x)j'(x)}(g^{-1})\mathrm{d}g
$$

of $U(N)$ lattice gauge theory.

4.1. The Gram matrix.
In order to calculate Weingarten integrals of the adjoint representaiton of $U(N)$, we first need to solve the basic problem of invariant theory for this representation. A partial solution to this problem is well-known, and part of a classical circle of ideas, commonly known as Schur-Weyl duality, which relate the representation theory of $U(N)$ to representations of the symmetric groups $S(d)$, $d \in \mathbb{N}$. In particular, it is known that, after identifying $(\mathrm{End}\,\mathcal{H})^{\otimes d}$ with $\mathrm{End}\,\mathcal{H}^{\otimes d}$, the space of $U(N)$-invariants is spanned by the operators which act by permuting tensor factors,

$$
A_\pi v_1 \otimes \cdots \otimes v_d = v_{\pi(1)} \otimes \cdots \otimes v_{\pi(d)}, \quad \pi \in S(d).
$$

Moreover, it is not difficult to compute the scalar product of any two of these operators: given $\rho, \sigma \in S(d)$, one finds that

$$
\langle A_\rho, A_\sigma \rangle = N^{\#\mathrm{cycles}(\rho^{-1}\sigma)},
$$

where $\#\mathrm{cycles}(\pi)$ is the number of factors in any factorization of π into disjoint cyclic permutations, so that the Gram matrix of these invariants is the $d! \times d!$ matrix

$$
\mathbf{A}^*\mathbf{A} = \left[N^{\#\mathrm{cycles}(\rho^{-1}\sigma)} \right]_{\rho,\sigma \in S(d)}.
$$

The reason we refer to this as a partial solution to the basic problem of invariant theory for the adjoint representation of $U(N)$ is that, although $\{A_\pi : \pi \in S(d)\}$ is a spanning set of invariants, it is only a basis in the *stable range*, where $1 \leq d \leq N$. In the unstable range, $d > N$, the operators A_π are linearly dependent, and their Gram matrix is singular. A satisfactory patch for this issue was found relatively recently by Baik and Rains [BR01], who showed that $\{A_\pi : \pi \in S_N(d)\}$ is always a basis, where $S_N(d) \subseteq S(d)$

is the set of permutations of $[d]$ with no decreasing subsequence of length $N+1$. Thus, the Gram matrix which needs to be inverted in order to calculate the degree d Weingarten integrals of the adjoint representation is actually

$$
\mathbf{A}^*\mathbf{A} = \left[N^{\#\mathrm{cycles}(\rho^{-1}\sigma)} \right]_{\rho,\sigma \in S_N(d)}.
$$

In the unstable range, the Gram matrix $\mathbf{A}^*\mathbf{A}$ must be computed numerically, but in the stable range we can view N as a parameter, so that the Weingarten matrix $\mathbf{W} = (\mathbf{A}^*\mathbf{A})^{-1}$ is a $d! \times d!$ matrix whose entries are rational functions of N. To get a handle on what these functions might be, it turns out to be a good idea to reinterpret the Gram matrix from the viewpoint of geometric group theory. More precisely, let us identify $S(d)$ with its (right) Cayley graph as generated by the conjugacy class of transpositions; then, the geodesic distance between permutations $\rho, \sigma \in S(d)$ is given by $|\rho^{-1}\sigma|$, where

$$
|\pi| = d - \#\mathrm{cycles}(\pi)
$$

is the word norm corresponding to the generating set of transpositions. Let q be a complex parameter, and consider the $d! \times d!$ matrix

$$
\Gamma = \left[\begin{matrix} & \vdots & \\ \cdots & q^{|\rho^{-1}\sigma|} & \cdots \\ & \vdots & \end{matrix} \right]_{\rho,\sigma \in S(d)},
$$

the q-distance matrix of the symmetric group $S(d)$. The q-distance matrix Γ is a deformation of the Gram matrix $\mathbf{A}^*\mathbf{A}$ — to recover the latter from the former, simply multiply by q^{-d} and then set $q = \frac{1}{N}$.

Thus, the problem we face is that of understanding the q-distance matrix of the symmetric group sufficiently well that we can invert it. This may be addressed via harmonic analysis on $S(d)$. The basic observation is that Γ is the matrix of the group algebra element

$$
\gamma = \sum_{\pi \in S(d)} q^{|\pi|} \pi
$$

acting in the right regular representation of $\mathbb{C}S(d)$. Moreover, γ is a central element in $S(d)$: in fact, we have

$$
\gamma = \sum_{r=0}^{d-1} q^r L_r,
$$

where L_r is the sum of all points on the sphere of radius r centered at the identity permutation $\iota \in S(d)$, or equivalently the sum of all permutations on the rth level of the Cayley graph. Clearly, every such sphere/level is a disjoint union of conjugacy classes. The plan is thus to take the Fourier transform of γ, i.e., its image under the algebra isomorphism

$$
\mathcal{F}: \mathbb{C}S(d) \longrightarrow \bigoplus_{\lambda \vdash d} \mathrm{End}\, V^\lambda, \tag{5}
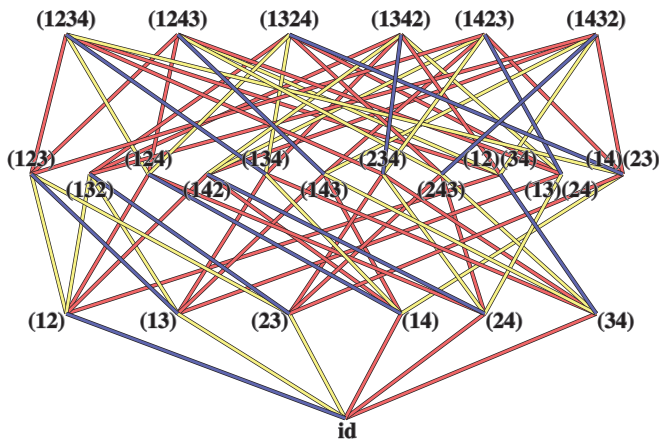$$

Figure 1. Biane-Stanley labeling of $S(4)$.

where (V^λ, R^λ) is the irreducible representation of $S(d)$ indexed by a given Young diagram λ with d cells, and

$$\mathcal{F}(a) = \bigoplus_{\lambda \vdash d} R^\lambda(a), \quad a \in \mathbb{C}S(d).$$

Since $\gamma \in \mathbb{C}S(d)$ is central, Schur's Lemma guarantees that $\mathcal{F}(\gamma)$ will be a direct sum of scalar operators, which can then easily be inverted. In particular, the computation reduces to calculating the Fourier transforms of the levels L_r of the Cayley graph.

The computation of the Fourier transform of L_r rests on a pair of remarkable discoveries in algebraic combinatorics made by the Lithuanian physicist Algimantas Adolfas Jucys (not to be confused with his father, the Lithuanian physicist Adolfas Jucys). The first of Jucys' discoveries is a unique factorization theorem for permutations. Let us call a factorization

$$\pi = (i_1 \ j_1) \dots (i_r \ j_r)$$

of a permutation $\pi \in S(d)$ into transpositions $(i \ j)$, where $1 \le i < j \le d$, a *strictly monotone factorization* if $j_1 < \cdots < j_r$.

Theorem 4.1. *Every permutation $\pi \in S(d)$ admits a unique strictly monotone factorization, and the number of factors in this factorization is $|\pi|$.*

This result may be visualized as follows. Let us mark each edge of the Cayley graph of $S(d)$ corresponding to the transposition $(i \ j)$ with j, the larger of the two symbols it interchanges. We call this the *Biane-Stanley labeling* of the symmetric group, since a version of it was considered first by Stanley and later by Biane in connection with the combinatorics of noncrossing partitions. Figure 1 depicts the Biane-Stanley labeling of $S(4)$, with 2-edges drawn in blue, 3-edges in yellow, and 4-edges in red. Call a walk on $S(d)$ a *strictly monotone walk* if the labels of the edges it traverse form a strictly increasing sequence. Jucys' result says that if we trace out all strictly monotone walks on $S(d)$ issuing from the identity permutation ι, we get a presentation of the symmetric group as a starlike tree.

Jucys' result gives us a new combinatorial description of the sphere L_r: it is the set of all permutations admitting a strictly monotone factorization of length r, i.e., the set of all points at distance r from ι on the Jucys tree. This in turn gives us a new algebraic description of L_r: it may be written as

$$L_r = e_r(J_1, \dots, J_d),$$

where

$$e_r(x_1, \dots, x_d) = \sum_{\substack{j \in \text{Fun}(r,d) \\ i \text{ strictly increasing}}} x_{j(1)} \dots x_{j(d)}$$

is the elementary symmetric polynomial of degree r, and $J_1, \dots, J_d \in \mathbb{C}S(d)$ are the transposition sums

$$J_j = \sum_{i < j}(i \ j), \quad 1 \le j \le d.$$

These sums are nowadays known as the *Jucys-Murphy* elements of $S(d)$. Although they are clearly non-central, it is not difficult to see that they commute with one another; in fact, they generate a maximal abelian subalgebra of $\mathbb{C}S(d)$ known as the *Gelfand-Tsetlin subalgebra*, whose role in the representation theory of $S(d)$ is analogous to the role of maximal tori in Lie theory [OV96].

This brings us to Jucys' second discovery. First, according to a classical result of Newton, the elementary symmetric polynomials are algebraically independent and generate the ring of symmetric polynomials. Thus, $f(J_1, \dots, J_d)$ lies in the center of $S(d)$ for any symmetric polynomial f, hence $f(J_1, \dots, J_d)$ acts as a scalar operator in any irreducible representation (V^λ, R^λ) of $S(d)$. What is its eigenvalue? This question was answered by Jucys in terms of the so-called "contents" of Young diagrams: if $\square \in \lambda$ is a cell of the diagram λ, drawn according to the English convention, its content $c(\square)$ is simply its column index minus its row index.

Theorem 4.2. *For any symmetric polynomial f and any Young diagram $\lambda \vdash d$, we have*

$$R^\lambda(f(J_1, \dots, J_d)) = \omega^\lambda(f)I_{V^\lambda},$$

where

$$\omega^\lambda(f) = f(c(\square) : \square \in \lambda)$$

is the evaluation of f on the multiset of contents of λ and I_{V^λ} is the identity operator in $\text{End } V^\lambda$.

The above results allow us to compute the Fourier transform of γ: by Jucys' first theorem, letting $\iota \in S(d)$ denote the identity permutation, we have the factorization,

$$\gamma = \sum_{r=0}^{d} q^r e_r(J_1, \dots, J_d) = \prod_{k=1}^{d}(\iota + qJ_k),$$

and hence by Jucys' second theorem we have

$$\mathcal{F}(\gamma) = \bigoplus_{\lambda \vdash d} \omega^\lambda(\gamma)I_{V^\lambda},$$

where
$$\omega^\lambda(\gamma) = \prod_{\square \in \lambda} (1 + qc(\square)).$$

This leads immediately to the conclusion that $\gamma \in \mathbb{CS}(d)$ is invertible for $|q| < \frac{1}{d-1}$, and that the Fourier transform of its inverse is
$$\mathcal{F}(\gamma^{-1}) = \bigoplus_{\lambda \vdash d} \omega^\lambda(\gamma^{-1}) I_{V^\lambda},$$

where the eigenvalue of γ^{-1} acting in V^λ is
$$\omega^\lambda(\gamma^{-1}) = \prod_{\square \in \lambda} (1 + qc(\square))^{-1}$$
$$= \sum_{r=0}^{\infty} (-q)^r h_r(c(\square) : \square \in \lambda),$$

where
$$h_r(x_1, \dots, x_d) = \sum_{\substack{j \in \mathrm{Fun}(r,d) \\ i \text{ weakly increasing}}} x_{j(1)} \dots x_{j(d)}$$

is the complete homogeneous symmetric polynomial of degree r.

4.2. The Weingarten matrix.
The preceding Fourier analysis of the q-distance matrix of $S(d)$ allows us to make a number of powerful statements about the Weingarten matrix $\mathbf{W}$ of the $U(N)$-invariants $A_\pi \in \mathrm{End}\,\mathcal{H}^{\otimes d}$, in the stable range $1 \le d \le N$.

The first such statement says that we can calculate the entries of the $d! \times d!$ matrix $\mathbf{W}$ explicitly provided we have access to the character table of $S(d)$.

Theorem 4.3. *For any $\rho, \sigma \in S(d)$, we have that*
$$\mathbf{W}_{\rho\sigma} = \sum_{\lambda \vdash d} \frac{\chi^\lambda(\rho^{-1}\sigma)}{\prod_{\square \in \lambda}(N + c(\square))} \frac{\dim V^\lambda}{d!},$$

where χ^λ is the character of V^λ.

Note that, since $\chi^\lambda(\rho^{-1}\sigma)$ depends only on the cycle type α of the product $\rho^{-1}\sigma$, i.e., the Young diagram whose row lengths encode the lengths of the disjoint cycles of this permutation, the matrix entry $\mathbf{W}_{\rho\sigma}$ itself depends only on α. We may thus define
$$\mathrm{Wg}^{U(N)}(\alpha) := \mathbf{W}_{\rho\sigma},$$

this being a function on Young diagrams known, reasonably enough, as the *Weingarten function* of the unitary group $U(N)$. One also writes $\mathrm{Wg}^{U(N)}$ when it is convenient to view the Weingarten function as a central function on permutations.

Combining Theorem 4.3 with the Fundamental Theorem of Weingarten Calculus, we thus obtain the following summation formula for the Weingarten integrals of adjoint representation of $U(N)$, which are exactly the link integrals of $U(N)$ gauge theory.

Theorem 4.4. *For any $1 \le d \le N$ and any $i, j \in \mathrm{Fun}(d, N)$, we have*
$$I_{ij} = \sum_{\rho,\sigma \in S(d)} \delta_{i,i'\rho} \delta_{j,j'\sigma} \mathbf{W}_{\rho\sigma}.$$

To the best of our knowledge, this summation formula first appeared in a 1980 physics paper of Samuel [Sam80]; it was independently rediscovered by Collins in [Col03]. The fact that the formula is confined to the stable range $1 \le d \le N$ turns out to be a minor issue, and this restriction can be easily lifted ([CŚ06]).

A more serious limitation on the utility of Theorem 4.4 is the fact that the characters of $S(d)$ are not at all simple objects; in fact, it is a known theorem of complexity theory that the irreducible characters of the symmetric groups are computationally intractable. Luckily, for many purposes, in both mathematical physics and random matrix theory, it is sufficient to have an asymptotic estimate for I_{ij} giving its approximate value as $N \to \infty$. It turns out that the Fourier analysis of the q-distance matrix discussed above gives a complete $N \to \infty$ asymptotic expansion for the entries of $\mathbf{W}$.

Theorem 4.5. *In the stable range $1 \le d \le N$, we have*
$$\mathbf{W}_{\rho\sigma} = \frac{(-1)^{|\rho^{-1}\sigma|}}{N^{d+|\rho^{-1}\sigma|}} \sum_{k=0}^{\infty} \frac{\vec{W}_k(\rho,\sigma)}{N^{2k}},$$

where $\vec{W}_k(\rho, \sigma)$ is the number of weakly monotone walks on $S(d)$ from ρ to σ of length $|\rho^{-1}\sigma| + 2k$.

A weakly monotone walk on the Cayley graph of $S(d)$ is similar to the strictly monotone walks discussed above, the difference being that labels of the edges traversed are only required to form a weakly increasing sequence. Unlike strictly monotone walks, there exist arbitrarily long weakly monotone walks between any two permutations ρ and σ, though these must satisfy a parity constraint depending on whether $\rho^{-1}\sigma$ is an even or odd permutation; this is why the series in Theorem 4.5 is a power series in N^{-2}. Theorem 4.5 gives a precise combinatorial interpretation of the famous $1/N$ expansion in $U(N)$ lattice gauge theory, cf [CM09]. The observation that monotone walks on symmetric groups play the role of Feynman diagrams for Haar integrals on $U(N)$ was first made in [Nov10], and further developed in [MN13]. In particular, the number of weakly monotone geodesics between any pair of permutations may be computed in closed form, giving a very useful first order approximation to the entries of $\mathbf{W}$.

Theorem 4.6. *For any $\rho, \sigma \in S(d)$, we have*
$$\vec{W}_0(\rho,\sigma) = \prod_{i=1}^{\ell(\alpha)} \frac{1}{\alpha_i} \binom{2\alpha_i - 2}{\alpha_i},$$

where $\alpha \vdash d$ is the cycle type of $\rho^{-1}\sigma$.

Yet another ramification of the realization that monotone walks on S(d) are the Feynman diagrams for Haar integration on U(N) is a family of identities that play the role of Schwinger-Dyson "loop" equations, and recursively determine the Weingarten function. The loop equations for $\mathrm{Wg}^{\mathrm{U}(N)}$ were first obtained by Samuel [Sam80], and later rediscovered in [CM17], who used them to obtain estimates in the unstable range $d > N$.

5. Orthogonal and Symplectic Groups

In this section, we extend the Weingarten calculus for unitary groups in the previous section to orthogonal and symplectic groups. The theory was first considered in [CŚ06], and further developed with the use of harmonic analysis of symmetry groups in [CM09, Mat13]. Since the Weingarten calculus for O(N) and Sp(N) is parallel to U(N), we focus on stating the results.

5.1. Pairings and hyper-octahedral groups. We realize the (real) orthogonal group O(N) as the compact matrix group consisting of all $N \times N$ real orthogonal matrices g, that is $gg^{\mathrm{T}} = I_N$. We are interested in the expectation of monomials $r_{i(1)j(1)} r_{i(2)j(2)} \dots r_{i(k)j(k)}$ in matrix elements $r_{xy} = \langle e_x, g e_y \rangle$ if g is distributed with respect to the Haar probability dg on O(N).

Since two random orthogonal matrices g and $-g$ are distributed in the same law, the integral

$$\int_{\mathrm{O}(N)} r_{i(1)j(1)} r_{i(2)j(2)} \dots r_{i(k)j(k)} \, dg$$

$$= \int_{\mathrm{O}(N)} (-r_{i(1)j(1)})(-r_{i(2)j(2)}) \dots (-r_{i(k)j(k)}) \, dg$$

vanishes if k is odd, so we consider only even-degree moments.

To do that, we introduce the notion of pairings and hyper-octahedral groups. Let $\mathcal{M}_{2d}$ be the set of all *pairings* of $\{1, 2, \dots, 2d\}$, that is, set partitions of $\{1, 2, \dots, 2d\}$ whose blocks are size two. Each pairing σ can be expressed in the form $\sigma = \{\{\sigma(1), \sigma(2)\}, \{\sigma(3), \sigma(4)\}, \dots, \{\sigma(2d-1), \sigma(2d)\}\}$, where $\sigma(1), \sigma(2), \dots, \sigma(2d)$ is a permutation of $1, 2, \dots, 2n$. We often write it in the condition

$$\sigma(2x-1) < \sigma(2x) \quad (1 \le x \le d), \tag{6}$$
$$1 = \sigma(1) < \sigma(3) < \dots < \sigma(2d-1),$$

and identify it with a permutation expressed in the same symbol σ in S_{2d}. Namely, we regard $\mathcal{M}_{2d}$ as a subset of S_{2d}. For example, a pairing $\{\{1, 5\}, \{2, 8\}, \{3, 4\}, \{6, 7\}\}$ is identified with the permutation $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 2 & 8 & 3 & 4 & 6 & 7 \end{smallmatrix}\right)$ in S_8.

Let H_d be the subgroup of S_{2d} generated by elements $(2x-1, 2x)$ with $1 \le x \le d$ and $(2x-1, 2y-1)(2x, 2y)$ with $1 \le x < y \le d$, where (p, q) stands for the transposition between p and q. We call it the *hyper-octahedral group* of degree d. The set $\mathcal{M}_{2d}$, which is regarded as a subset of S_{2d},

forms a complete set of representatives of left cosets $\sigma \mathrm{H}_d$ in S_{2d}.

Furthermore, in order to distinguish double cosets $\mathrm{H}_d \sigma \mathrm{H}_d$, we consider an undirected multigraph $\Gamma(\sigma)$ for each $\sigma \in \mathrm{S}_{2d}$ as follows. The vertex set of $\Gamma(\sigma)$ is $\{1, 2, \dots, 2d\}$, and the edge set consists of $\{\{2x-1, 2x\} \mid 1 \le x \le d\}$ and $\{\{\sigma(2x-1), \sigma(2x)\} \mid 1 \le x \le d\}$. Each vertex lies on exactly two edges. Then connected components of $\Gamma(\sigma)$ are cycles of even lengths $2\mu_1, 2\mu_2, \dots, 2\mu_l$, where we arrange them with $\mu_1 \ge \mu_2 \ge \dots \ge \mu_l \ge 1$. We call the (integer) partition $\mu = (\mu_1, \mu_2, \dots, \mu_l)$ of d the *coset-type* of σ. For example, for a permutation $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 2 & 8 & 4 & 3 & 6 & 7 \end{smallmatrix}\right)$, one connected component of $\Gamma(\sigma)$ has six vertices $1, 5, 6, 7, 8, 2$, and another component has two vertices $3, 4$; so its coset-type is $\mu = (3, 1)$. It is known that two permutations σ, τ in S_{2d} have the same coset-type if and only if they belong to the same double coset of H_d in S_{2d}, i.e., $\mathrm{H}_d \sigma \mathrm{H}_d = \mathrm{H}_d \tau \mathrm{H}_d$. The length $\kappa(\sigma)$ of the coset-type of $\sigma \in \mathrm{S}_{2d}$ is important. Equivalently, it is the number of connected components in the graph $\Gamma(\sigma)$.

5.2. Weingarten formula for orthogonal groups. Now we give Weingarten formula for the orthogonal group O(N). For any $i, j \in \mathrm{Fun}(2d, N)$, we have the formula

$$\int_{\mathrm{O}(N)} r_{i(1)j(1)} r_{i(2)j(2)} \dots r_{i(2d)j(2d)} \, dg$$

$$= \sum_{\sigma \in \mathcal{M}_{2d}} \sum_{\tau \in \mathcal{M}_{2n}} \Delta_\sigma(i) \Delta_\tau(j) \, \mathrm{Wg}^{\mathrm{O}(N)}(\sigma^{-1}\tau), \tag{7}$$

where $\Delta_\sigma(i)$ is, by definition, equal to 1 if $i(a) = i(b)$ for every pair $\{a, b\}$ in σ; to zero otherwise. We here skip a detailed definition of $\mathrm{Wg}^{\mathrm{O}(N)}$, which can be obtained by the same argument as in the case of unitary groups, but we look at a few examples first. For each permutation σ, the value $\mathrm{Wg}^{\mathrm{O}(N)}(\sigma)$ depends on only its coset-type. We denote by σ_μ a specific permutation with coset-type μ. Then we may see that

$$\mathrm{Wg}^{\mathrm{O}(N)}(\sigma_1) = \frac{1}{N}, \tag{8}$$

$$\mathrm{Wg}^{\mathrm{O}(N)}(\sigma_{1,1}) = \frac{N+1}{N(N-1)(N+2)}, \tag{9}$$

$$\mathrm{Wg}^{\mathrm{O}(N)}(\sigma_2) = \frac{-1}{N(N-1)(N+2)}. \tag{10}$$

Let us see an application for formula (7). Consider the functions $i, j \in \mathrm{Fun}(4, N)$ with values $(i(x))_{x=1}^4 = (1, 1, 2, 2)$ and $(j(x))_{x=1}^4 = (2, 3, 2, 3)$. Then $\Delta_\sigma(i) = 1$ only if $\sigma = \{\{1, 2\}, \{3, 4\}\}$; $\Delta_\tau(j) = 1$ only if $\tau = \{\{1, 3\}, \{2, 4\}\}$. When we regard these σ, τ as permutations, the coset-type of $\sigma^{-1}\tau$ is the same with that of σ_2. Thus, we obtain the integral value

$$\int_{\mathrm{O}(N)} r_{12} r_{13} r_{22} r_{23} \, dg = \mathrm{Wg}^{\mathrm{O}(N)}(\sigma_2) = \frac{-1}{N(N-1)(N+2)}.$$

The discussion of orthogonal Weingarten functions can be almost parallel to that of unitary cases, but in a slightly more complicated form. For example, the counterpart of the $1/N$ expansion of the unitary Weingarten function is as follows: for any $1 \le d \le \frac{N+1}{2}$ and any $\alpha \vdash d$, we have

$$\mathrm{Wg}^{O(N)}(\sigma_\alpha) = \frac{(-1)^{d-\ell(\alpha)}}{N^{2d-\ell(\alpha)}} \sum_{k=0}^{\infty} (-1)^k \frac{\vec{W}_k'(\alpha)}{N^k},$$

where $\vec{W}_k'(\alpha)$ is a non-negative integer enumerating certain analogues of monotone walks on $\mathcal{M}_{2d}$.

5.3. Weingarten formula for symplectic groups.

Let $J = J_N$ be the $2N \times 2N$ skew symmetric matrix given by

$$J_N = \begin{pmatrix} O & I_N \\ -I_N & O \end{pmatrix}. \tag{11}$$

The (unitary) symplectic group $\mathrm{Sp}(N)$ is realized as $\mathrm{Sp}(N) = \{g \in \mathrm{U}(2N) \mid S^T J S = J\}$. This preserves the skew symmetric bilinear form on $\mathbb{C}^{2N}$ given by $\langle v, w \rangle_J = v^T J w$. If the collection $\{e_1, \ldots, e_{2N}\}$ is the standard basis of $\mathbb{C}^{2N}$, then it is immediate to see that

$$\langle e_x, e_y \rangle_J = \begin{cases} 1 & \text{if } y = x + N, \\ -1 & \text{if } x = y + N, \\ 0 & \text{otherwise.} \end{cases}$$

The Weingarten formula for $\mathrm{Sp}(N)$ is quite similar to $O(N)$ but we need to treat signatures carefully. Consider the integral $\int_{\mathrm{Sp}(N)} s_{i(1)j(1)} s_{i(2)j(2)} \cdots s_{i(k),j(k)} \, dg$ of matrix elements, where dg is the Haar probability on $\mathrm{Sp}(N)$. As for the orthogonal groups, this integral vanishes if k is odd. Here we use matrix elements s_{xy} of g rather than the value of the skew form $\langle e_x, ge_y \rangle_J$.

For each pairing $\sigma \in \mathcal{M}_{2d}$ and $i \in \mathrm{Fun}(2d, 2N)$ we define

$$\Delta_\sigma'(i) = \prod_{x=1}^{d} \langle e_{i\sigma(2x-1)}, e_{i\sigma(2x)} \rangle_J.$$

This Delta-symbol takes the value of 1, -1, or 0. Here we must watch the assumption (6); otherwise, the sign of this may be accidentally changed.

Now we provide the Weingarten formula for symplectic groups. For any $i, j \in \mathrm{Fun}(2d, 2N)$, we have

$$\int_{\mathrm{Sp}(N)} s_{i(1)j(1)} s_{i(2)j(2)} \cdots s_{i(2d)j(2d)} \, dg$$
$$= \sum_{\sigma \in \mathcal{M}_{2d}} \sum_{\tau \in \mathcal{M}_{2d}} \Delta_\sigma'(i) \Delta_\tau'(j) \mathrm{Wg}^{\mathrm{Sp}(N)}(\sigma^{-1}\tau). \tag{12}$$

Let us see an example for symplectic Weingarten formula (12). Consider the integral

$$\int_{\mathrm{Sp}(N)} s_{1,1} s_{2,N+2} s_{N+1,2} s_{N+2,N+1} \, dg,$$

so we apply (12) with $i = (1, 2, N+1, N+2)$ and $j = (1, N+2, 2, N+1)$. Then only parings $\sigma = \{\{1,3\},\{2,4\}\}$

and $\tau = \{\{1,4\},\{2,3\}\}$ contribute to the sum in (12), and we have $\Delta_\sigma'(i) = \langle e_1, e_{N+1} \rangle_J \langle e_2, e_{N+2} \rangle_J = +1$ and $\Delta_\tau'(j) = \langle e_1, e_{N+1} \rangle_J \langle e_{N+2}, e_2 \rangle_J = -1$. Moreover, the permutation $\sigma^{-1}\tau$ is $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{smallmatrix}\right)$, which is of sign -1 and of coset-type (2). In the present text, we do not give the definition of the symplectic Weingarten function, but such an observation show that the integral is equal to

$$\int_{\mathrm{Sp}(N)} s_{1,1} s_{2,N+2} s_{N+1,2} s_{N+2,N+1} \, dg$$
$$= \mathrm{Wg}^{\mathrm{Sp}(N)}(\sigma_2) = \frac{1}{4N(N-1)(2N+1)}.$$

5.4. Circular ensembles.

In random matrix theory, not only classical compact groups $\mathrm{U}(N), O(N), \mathrm{Sp}(N)$ but also circular ensembles are well studied. The three main examples are circular orthogonal/unitary/symplectic ensembles (COE/CUE/CSE). In this subsection, we will follow the symbols of Random Matrix Theory and regard random matrices as matrix-valued random maps, and write integrals $\int \cdots dg$ in the form of expectation values $\mathbb{E}[\cdots]$.

The CUE matrix is nothing but the Haar-distributed unitary matrix, the Weingartn calculus for which is already given in the previous section. Let U and $\tilde{U}$ be two CUE matrices of dimension N and $2N$, respectively. Then the COE matrix $V = (v_{xy})_{x,y=1}^N$ and CSE matrix $\tilde{H} = (\tilde{h}_{xy})_{x,y=1}^{2N}$ are determined by $V = UU^T$ and $\tilde{H} = \tilde{U}J\tilde{U}^T J^T$, with the matrix J defined in (11), respectively. However, for a technical reason, we consider a modified CSE matrix $H = \tilde{U}J\tilde{U}^T$ rather than $\tilde{H} = HJ^T$.

The Weingarten formulas for them are given as follows. We denote by $\mathbb{E}$ the corresponding expectation for each random matrix. For any $i \in \mathrm{Fun}(2m, N)$ and $j \in \mathrm{Fun}(2n, N)$, we have the formula for the COE

$$\mathbb{E}\big[v_{i(1)i(2)} v_{i(3)i(4)} \cdots v_{i(2m-1)i(2m)} \overline{v_{j(1)j(2)} v_{j(3)j(4)} \cdots v_{j(2n-1)j(2n)}} \big]$$
$$= \delta_{mn} \sum_{\sigma \in S_{2n}} \delta_{i,j\sigma} \, \mathrm{Wg}^O(\sigma; N+1). \tag{13}$$

Similarly, for $i \in \mathrm{Fun}(2m, 2N)$ and $j \in \mathrm{Fun}(2n, 2N)$, we have the formula for the CSE

$$\mathbb{E}\big[h_{i(1)i(2)} h_{i(3)i(4)} \cdots h_{i(2m-1)i(2m)} \overline{h_{j(1)j(2)} h_{j(3)j(4)} \cdots h_{j(2n-1)j(2n)}} \big]$$
$$= \delta_{mn} \sum_{\sigma \in S_{2n}} \delta_{i,j\sigma}.$$

Here $\mathrm{Wg}^O(\sigma; z)$ and $\mathrm{Wg}^{\mathrm{Sp}}(\sigma; z)$ are the rational function in z, obtained N by a complex number z for $\mathrm{Wg}^{O(N)}(\sigma)$ and $\mathrm{Wg}^{\mathrm{Sp}(N)}(\sigma)$, respectively.

Surprisingly, when we think of COE and CSE, we do not need any new Weingarten function, but a different

parameter of the orthogonal/symplectic Weingarten functions suffice.

The COE and CSE are deeply related to compact symmetric spaces U(N)/O(N) and U(2N)/Sp(N), respectively. For other kinds of compact symmetric spaces, with corresponding various random matrices, similar rich Weingarten formulas are known.

Historically, the formula (13) first appeared in [BB96] without proof. Mathematical treatment for COE and other compact symmetric spaces were done in [Mat12, Mat13].

6. Conclusion and Outlook

In this article, we have only scratched the surface of Weingarten calculus, both in terms of theory and applications.

On the theoretical side, the results we have presented for integration on U(N), and only touched on for O(N) and Sp(N), can be rendered in much more detail and admit many powerful generalizations which we have not discussed here. Moreover, the entire apparatus can be developed in the context of compact symmetric spaces and compact quantum groups, where the results are just as rich and varied as for classical compact topological groups. We touched on Weingarten calculus for symmetric spaces when discussing circular ensembles of random matrices above, and here we will briefly indicate the situation for compact quantum groups. Roughly speaking, compact quantum groups are noncommutative C^*-algebras obtained from the C^*-algebras of classical compact topological groups by suppressing commutativity. They enjoy the same key properties as the function algebras of classical compact groups, namely they satisfy a Peter-Weyl theorem, a Tannaka-Krein duality, they admit a finite left and right invariant Haar measure, and all their irreducible representations are of finite dimension. The theory was created by Woronowicz, who laid these foundations in a series of landmark papers. A version of the Weingarten calculus for the computation of Haar integrals on compact quantum group was derived in [BC07], as an extension of the works of [Col03], and has since found many applications in functional analysis and operator algebras. Our forthcoming monograph gives the first pedagogical account of this new theory.

Concerning applications of the Weingarten calculus, there are many. Historically, one of the first applications of Weingarten calculus is a systematic approach to asymptotic freeness of random matrices, a phenomenon discovered by Voiculescu in the context of free probability theory, see, e.g., [VDN92]. Roughly speaking, free probability theory is a noncommutative probability theory in which the notion of independence is based on the free product of algebras, as opposed to the tensor product, which gives classical independence. This notion arises naturally in the study of certain von Neumann algebras, but Voiculescu

discovered that large, classically independent random matrices in fact approximate free random variables. We refer to [MS17] or [Col22] for references. This fact is enormously useful in random matrix theory, as it allows the machinery of free probability theory to be harnessed in order to study the asymptotic spectral behavior of families of large random matrices. Initially, the connection between random matrices and free probability was only applicable to global observables of the spectrum, such as expectations of traces of powers as discussed earlier. It turns out that, when the machinery of Weingarten calculus is brought into the picture, it becomes possible to amplify this connection to *strong asymptotic freeness*, which enables the use of free probability methods to handle non-global observables, such as the operator norm of random matrices. It turns out that this boost is precisely what is needed to bring the tools of random matrix theory and free probability to bear on theoretical problems in quantum information theory ([CN16]).

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