Gerrymandering, Sandwiches, and Topology

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ABSTRACT. We describe some applications of measure partitioning theorems, which extend the ham sandwich theorem, to draw biased political maps. The results we use are at the crossroads of metric geometry and algebraic topology.

In May, I gave a talk about fair division problems in a seminar. These are the kind of mathematical results that you would use if, for example, you wanted to divide rent fairly with your roommates, split a cake so that no one is envious of another person's piece, or distribute a stolen necklace among your fellow thieves. I was asked if those results could be used to "solve gerrymandering." I was surprised that a direct application would not help fix this problem, but rather make it worse. In particular, one would be able to draw extremely gerrymandered maps without using strange shapes for the districts, which goes against the intuition in this subject.

Gerrymandering is the practice of drawing political maps to gain an advantage. There is evidence of gerrymandered maps all over the world, and it has been a reason for heated debate in the United States for over 200 years. Suppose that you, a cartographer consulting for the government, are tasked with dividing the country into districts. Each district will have a representative. Every person will choose a color, blue or red, and the color of the representative of each district will be determined by the majority of the votes there.

If 57 percent of the population voted blue and 43 percent voted red, it would seem fair that roughly 57 percent of the representatives are blue, and 43 percent are red. However, a well-drawn map can drastically change this, as in Figure 1. The person drawing the map has much more power on the distribution of representatives than it may seem at first sight.

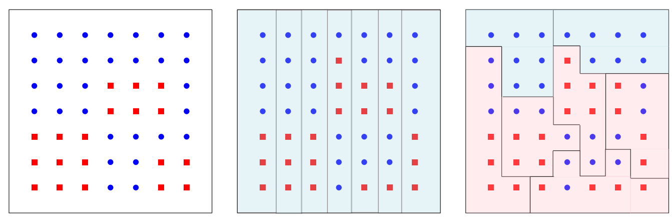

Figure 1. Two different divisions of the same electoral map, where 57 percent of the votes are blue and 43 percent are red, into seven districts. In the first division blue wins 100 percent of the districts. In the second division blue wins 28 percent of the districts and red wins 72 percent.

An ill-intentioned cartographer has two tools at his disposal: packing and cracking. Packing refers to concentrating a group in a single district where they win by a large margin, thereby minimizing the impact of their votes. Cracking refers to dispersing a group across many districts, thereby diluting the impact of their votes. As you may imagine, very biased maps often end up having districts with unusual shapes to make the most of these tools. Indeed, this is where the name gerrymandering comes from, as one of the districts drawn in the redistricting map of Massachusetts signed by Elbridge Gerry in 1812 was said to resemble a salamander. Figure 2 shows an example of a district with an odd shape.

There has been a lot of effort to use mathematics to understand and detect gerrymandering. One instance is the group led by Moon Duchin at Tufts University [3]. Many approaches rely on finding oddly shaped districts or counting voting efficiency. It is not an easy task, especially since it is sometimes difficult to tell apart intentional partisan gerrymandering and accidental gerrymandering.

Let's see what we could do if we took the job of the biased cartographer and decided to use mathematics to gerrymander on purpose. What is the worst we could do

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See Opinion piece "A Formula Goes to Court: Artisan Gerrymandering and the Efficiency Gap" (page 1020).

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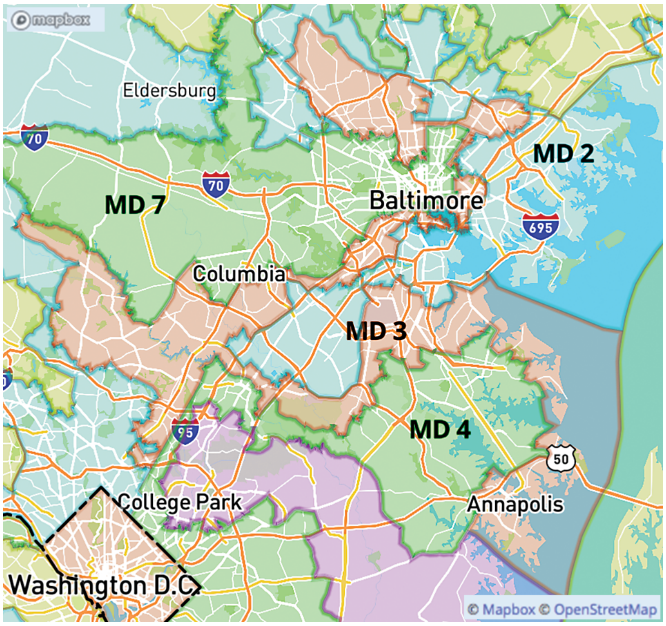


Figure 2. Maryland's third congressional district (in brown) has an odd shape.

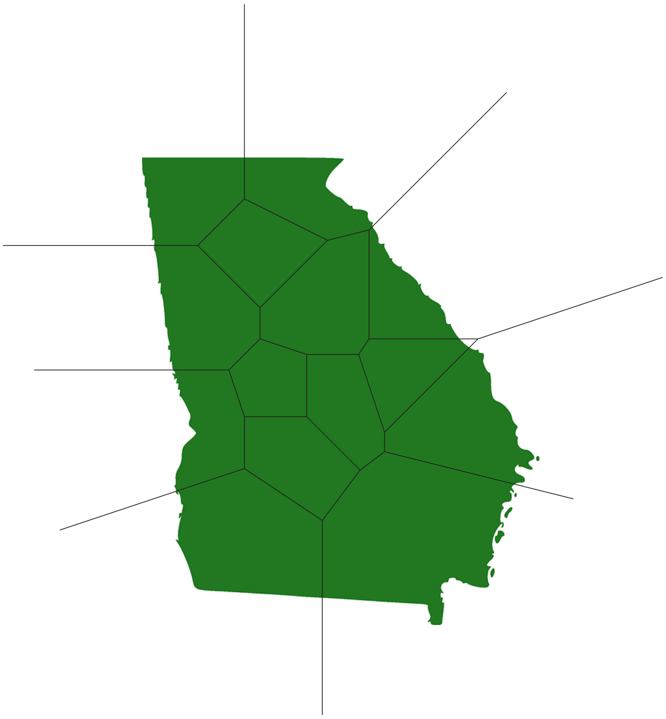


Figure 3. A map of the state of Georgia divided using fourteen convex sets.

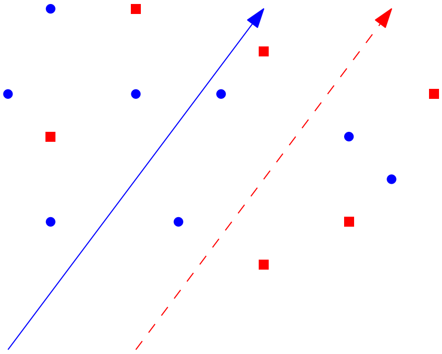


Figure 4. Two parallel lines with the same direction splitting their colored point sets.

without using strange shapes for the districts? We will ask the following of our map:

- The districts will have *convex* shapes, up to the boundary of the map. In other words, if two points are inside a district, the straight segment between them is also there. This prevents any dents or holes in the districts. Figure 3 shows an example with the state of Georgia.
- Each district will hold the same number of voters, in an effort to promote "equal representation."

With these conditions, what is the worst we could do? If we are inclined to benefit the party that already has the majority of the total votes, I claim that we can always get them to win *every single district.*

The tool we will use for this is a generalization of a particularly beautiful theorem in discrete geometry, called the *ham sandwich theorem.* The ham sandwich theorem was conjectured by Steinhaus and subsequently proved by Banach in 1938. This result is one of the first applications of equivariant algebraic topology to metric geometry. In formal terms, equivariant topology is the branch of mathematics that studies continuous functions between topological spaces that preserve some kind of symmetry; certainly something that seems far detached from our map-drawing goals. For two dimensions, it says the following.

Theorem 1. *Given two finite sets of points in the plane, blue and red, both with an even number of points and such that no three colored points are collinear, there is a line that splits simultaneously both colors in half.*

The reason for the name is the interpretation where each color represents an ingredient on a table. Then, for our two ingredients (say, ham and cheese) we can always find a straight line that leaves exactly half of each ingredient on each side.

To prove this result, draw two parallel lines. One will be red, splitting the red points in half, and one will be blue, splitting the blue points in half, as in Figure 4. If we are lucky, they are the same line and we are done. If not, assign a direction to them so that they are pointing to the same side, and start rotating them clockwise, always keeping them parallel. It turns out that we can move them as we rotate in order for each line to split its corresponding color in half all the time. Moreover, the movement can be continuous. Once you've given a half-turn, you will end up with the same two lines that you started with, but

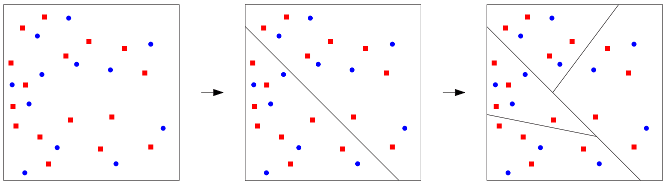
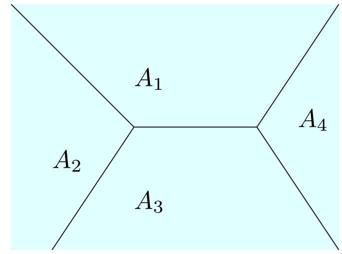


Figure 5. After two steps, we have a division into four convex districts where the same party wins all.



$$f_1(P) = \left(\mu_1(A_1) - \tfrac{1}{4}, \mu_1(A_2) - \tfrac{1}{4}, \mu_1(A_3) - \tfrac{1}{4}, \mu_1(A_4) - \tfrac{1}{4}\right)$$

Figure 6. We can repeat the argument with $n = 4$. Notice that if $\mu_1(\mathbb{R}^2) = 1$, then $\text{image}(f_1) \cong \mathbb{R}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$.

pointing in the opposite direction. If at the beginning a person walking along the red line had the blue line on his left, at the end he has the blue line on his right. Thus, at some point the two lines had to coincide, giving us the desired line. If the resulting line is a degenerate case that goes through a point, it has to go through two points of the same color. A small perturbation fixes this problem.

Let's see how we can use the ham sandwich theorem to draw our map. We start with a particular case, and suppose that the number of districts we want to produce is a power of two. Then, we can find a straight line that splits both colors in half. We can repeat the argument for each new region. We can continue this way, and in each new region we can split both colors in half simultaneously. After doing this k times, we will have 2^k districts, all convex and all with the same portion of the blue voters and the same portion of the red voters, as in Figure 5. Whichever color had the majority of the votes will have the majority in each district, as we wanted.

Unfortunately, since the number of districts in most states is not a power of two, this method of repeatedly halving the population will not work. Fortunately, we can find similar biased maps, regardless of the number of congressional districts we want.

Theorem 2 (Bespamyatnikh, Kirkpatrick, Snoeyink 2000; Sakai 2002 [1], [4])**.** *Suppose we are given positive integers g, h, n, a set of gn red points in the plane, and a set of hn blue points in the plane, so that no three colored points lie in a line. Then, we can split the plane into n convex sets, each with exactly g red points and h blue points.*

The result above with $n = 14$ would give us a congressional map for Georgia, perhaps as in Figure 3, using simple shapes that greatly benefit whichever party has won the total vote. Let us describe the general framework to prove a result such as Theorem 2. This is a standard approach often called the *test map scheme*. One of the best introductions to this technique remains the book by Jiří Matoušek [2].

First, we assume that instead of red and blue points, we are given two smooth probability measures μ_1, μ_2. The result with point sets can be recovered via approximation results (this is not always trivial). Once we are set with a family of partitions of the plane into 14 parts that we want to use, we can parametrize it with a space X. Now we can form a function $f_1 : X \to \mathbb{R}^{13}$, that depends on μ_1, as follows. Given a partition $P = (A_1, \dots, A_{14}) \in X$, we define

$$f_1(P) = \left(\mu_1(A_1) - \frac{1}{14}, \dots, \mu_1(A_{14}) - \frac{1}{14}\right).$$

Notice that the image is contained in a 13-dimensional subspace of $\mathbb{R}^{14}$ because the sum of the coordinates is always zero. A similar example is shown in Figure 6. We repeat the construction with μ_2, and consider $f = (f_1, f_2) : X \to \mathbb{R}^{2 \cdot 13}$. The conditions on the measures make f continuous. We are looking for an partition P such that $f(P) = \bar{0}$. If it does not exist, we can reduce the dimension of the image and construct a function to a $(2 \cdot 13 - 1)$-dimensional sphere $\tilde{f} : X \to S^{2 \cdot 13 - 1}$ such that $\tilde{f}(P) = \frac{f(P)}{||f(P)||}$.

Moreover, there is a natural action of the symmetric group S_{14} in both spaces, which only permutes the names of the districts. With this action, we have that $\tilde{f}$ is equivariant; i.e. $\tilde{f}(gP) = g\tilde{f}(P)$ for all $g \in S_{14}$. The question now becomes one of equivariant topology: *Prove that there is no S_{14}-equivariant continuous map*

$$\tilde{f} : X \to_{S_{14}} S^{2 \cdot 13 - 1}.$$

Sometimes factoring arguments and other tricks reduce the amount of algebraic topology needed. The proof becomes a tug of war where one side consists of the construction and paramentrization of X and factorization tricks, and in the other side you have the topological machinery needed to solve the resulting problem. For instance, Theorem 2 and the ham-sandwich theorem can be proven by clever applications of the mean value theorem. They both have generalizations in higher dimensions, where we have points of as many colors as the dimension, whose proofs rely on stronger results from algebraic topology, such as the Borsuk-Ulam theorem and Dold's theorem.

Theorem 3 (Dold 1983)**.** *Given a finite group G with $|G| > 1$, if X and Y are two paracompact spaces with free actions of G, X is n-connected and Y is at most n-dimensional, there is no G-equivariant map $f : X \to_G Y$.*

Other extensions exist if you want to divide the plane (or a convex object) into convex pieces that are equal in a different light. For example, let us consider partitions of an equilateral triangle into five convex pieces. However, instead of requiring that each set has the same number

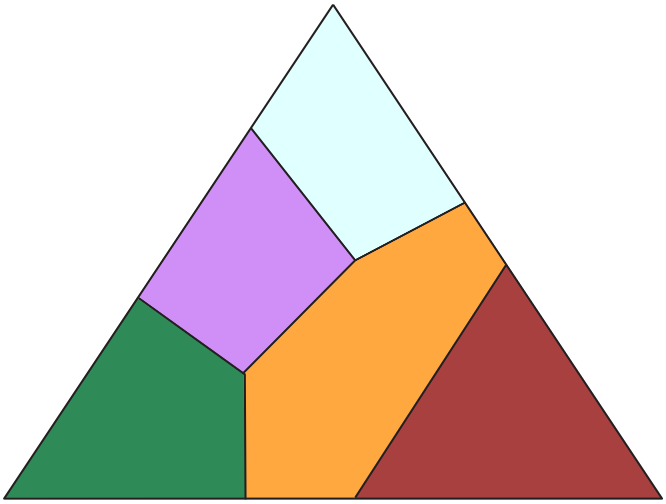


Figure 7. Trying to find partitions with parts of equal area and equal perimeter breaks most factorization tricks.

of blue and red points, is it possible for the five pieces to have equal area and equal perimeter? An attempt is shown in Figure 7, but it is no easy task.

This seemingly innocent question was first asked by Nandakumar and Rao. Direct applications of Dold's theorem fail, so one has to dive deeper into topological methods. The answer to this problem is positive, and has far-reaching generalizations, best explained in an expository article by Günter M. Ziegler [5]. Those results point out the fact that if there is an accepted formula to measure gerrymandering, it might still be possible to subdivide a map into convex pieces where each part has the same proportion of each colored set and all yield the same result under the formula (we are no longer requiring all districts to hold the same population).

In other words, even with strong conditions on the shape of the districts, gerrymandering can be done. An analysis of an electoral map should not be based solely on the geometry of the districts. In particular, you should be wary if the person drawing congressional maps knows his share of algebraic topology.

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EDITOR'S NOTE. Applications of similar techniques to problems of optimization and mathematical economics is one area of focus for the current MSRI program "Geometric and Topological Combinatorics" `www.msri.org/programs/309`.

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Pablo Soberón's research lies in the connection of topology, linear algebra, and combinatorics. In particular, he is interested in discrete geometry and combinatorial topology.

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