JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 3, July 1995

THE $K(\pi, 1)$-PROBLEM FOR HYPERPLANE COMPLEMENTS ASSOCIATED TO INFINITE REFLECTION GROUPS

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INTRODUCTION

We begin by recalling some well-known facts. The natural action of the symmetric group S_n on $\mathbb{R}^n$ can be viewed as a group generated by reflections. The reflections in S_n are the orthogonal reflections across the hyperplanes $H_{ij} = \{x \in \mathbb{R}^n | x_i = x_j\}$, $1 \leq i < j \leq n$. The S_n-action extends to $\mathbb{C}^n$ $(= \mathbb{R}^n \otimes \mathbb{C})$ and the set of points with nontrivial isotropy group is $\bigcup H_{ij} \otimes \mathbb{C}$. Let M denote the quotient manifold,

$$M = (\mathbb{C}^n - \bigcup H_{ij} \otimes \mathbb{C})/S_n.$$

Classically, one knows that the fundamental group of M is the braid group B_n. In [FN], it is proved that M is an Eilenberg-MacLane space $K(B_n, 1)$.

These facts were generalized by Brieskorn [Bn] and Deligne [D] to finite reflection groups as follows. Associated to a Coxeter system (W, S) there is an Artin group A (or "generalized braid group") constructed by deleting some of the relations in the standard presentation for W. (This is explained in Section 1.) When the Coxeter group W is finite, we say A is of *finite type*. In this case, there is a (unique) representation of W as an orthogonal reflection group on $\mathbb{R}^n$ (with $n = \mathrm{Card}(S)$). Consider the induced W-action on $\mathbb{C}^n$ and the quotient manifold

$$M = (\mathbb{C}^n - \bigcup \text{ reflection hyperplanes})/W.$$

In [Bn], Brieskorn proved that the fundamental group of M is A, and in [D], Deligne proved that M is an Eilenberg-MacLane space.

As explained below, there is a natural guess as to how this should generalize to infinite Coxeter groups. According to [L] this conjecture is due to Arnold, Pham and Thom. In this paper we shall prove this conjecture for almost all infinite Coxeter groups.

In order to formulate the conjecture we first need to explain the notion of a "linear reflection group" in the case when the group is infinite. A *linear reflection* on a real vector space V is a linear involution with fixed subspace a

Received by the editors December 3, 1993.

1991 *Mathematics Subject Classification*. Primary 51F15, 20F36.

The authors were partially supported by NSF grants and the Institute for Advanced Study during the preparation of this paper.

hyperplane. (Since there is no assumption of orthogonality, the reflection is not determined by its hyperplane.) Let $\overline{C}$ be a polyhedral cone of full dimension in V and let C denote its interior. Let $H_1, \cdots, H_k$ be the hyperplanes spanned by the codimension one faces of $\overline{C}$. For each i, let s_i be a linear reflection with fixed hyperplane H_i and let W be the subgroup of $GL(V)$ generated by $\{s_1, \cdots, s_k\}$. Then W is called a *linear reflection group* if $w(C) \cap C = \varnothing$ for all $w \in W$ with $w \neq 1$.

For each point x in $\overline{C}$ let W_x denote the sugroup of W generated by the reflections across the hyperplanes of $\overline{C}$ which contain x, and let

$$C^f = \{x \in \overline{C} \mid W_x \text{ is finite}\}.$$

Given a Coxeter system (W, S), let $\mathcal{S}^f$ be the poset of those subsets T of S such that the subgroup W_T generated by T is finite.

There is the following basic result of [V].

Theorem (Vinberg). *Let W be a linear reflection group on V and let*

$$\overline{I} = \bigcup_{w \in W} w\overline{C}.$$

Then the following statements hold.

(i) $(W, \{s_i\})$ *is a Coxeter system.*

(ii) $\overline{I}$ *is a convex cone.*

(iii) *The interior I of $\overline{I}$ is W-stable and W acts properly on I.*

(iv) $I \cap \overline{C} = C^f$

(v) $\mathcal{S}^f$ *can be identified with the opposite poset to the poset of faces of C^f.*

The cone $\overline{I}$ is called the *Tits cone.*

Tits proved that any Coxeter group W admits a faithful representation as a linear reflection group with $\overline{C}$ a simplicial cone (see [Bo, Ch. V, §4.4–4.6]). In general, however, as shown in [V], the cone $\overline{C}$ need not be simplicial and there may be many possible representations of W as a linear reflection group.

Infinite linear reflections groups are important in the theory of Kac-Moody Lie algebras. A central role is played by the domain Ω in $V \otimes \mathbb{C}$ defined by

$$\Omega = \{v \in V \otimes \mathbb{C} | \mathrm{Im}(v) \in I\} = V + iI.$$

By Vinberg's theorem, W acts properly on Ω and freely on ($\Omega - \bigcup$ reflection hyperplanes). Let

$$M = (\Omega - \bigcup \text{reflection hyperplanes})/W.$$

In various cases, the manifold M occurs in algebraic geometry as the complement of the discriminant of the semiuniversal deformation of a singularity.

In his thesis [L], H. van der Lek proved that the fundamental group of M is the Artin group A associated to W. (A special case had been proved earlier in [N].) The conjecture which we are interested in is the following.

Conjecture 1 (Arnold, Pham, Thom). *M is an Eilenberg-MacLane space.*

Some progess on this conjecture was made in [L].

Our approach to Conjecture 1 has three steps.

Step 1. There is a combinatorial reformulation of the conjecture which shows that its truth depends only on the Coxeter system (W, S) and not on its particular representation as a linear reflection group. This is explained in §2.

Step 2. Associated to the Artin group A there is a cetain simplicial complex Φ, which we define in §1 and call the "modified Deligne complex". Using the results of [D], it can be shown that Φ is simply connected and that Conjecture 1 is equivalent to the following.

Conjecture 2. Φ *is contractible.*

Step 3. The complex Φ can be given a piecewise Euclidean structure. When this structure is $CAT(0)$, in the sense of Gromov [G], then the contractibility of Φ follows. Since Φ is simply connected, this reduces to checking that the induced piecewise spherical metric on the link of each vertex is $CAT(1)$.

Step 1 is accomplished in §2 and Step 2 in §§1 and 3. The core of the paper lies in the work on Step 3 in §4.

Associated to any Coxeter system (W, S) is a "Coxeter complex" $\hat{\Sigma}$ on which W acts with "fundamental chamber" a simplex of dimension $n = \mathrm{Card}(S) - 1$. If W is finite, then $\hat{\Sigma}$ can be identified with the sphere $\mathbb{S}^n$, and we call $\hat{\Sigma}$ *spherical.* An analogous complex $\hat{\Phi}$, called the "Deligne complex" (since it was introduced in [D]), can be constructed for the associated Artin group A. The Deligne complex is "building-like" in the sense that it is made up of subcomplexes ("apartments") each isomorphic to $\hat{\Sigma}$. If A is of finite type, we say that $\hat{\Phi}$ is *spherical.*

In light of our definition of Ω in terms of the Tits cone, part (iv) of Vinberg's theorem indicates that we need a complex in which only finite subgroups of W appear as isotropy groups. We therefore consider a "modified Coxeter complex" Σ with isotropy groups W_T, $T \in \mathcal{S}^f$, and "fundamental chamber" K, which is the geometric realization of the derived complex of $\mathcal{S}^f$. The complex Φ which arises in Step 2 is the analoguous complex for A, with fundamental chamber K and with isotropy groups the sub-Artin groups A_T of finite type. Φ is "building-like" with "apartments" isomorphic to Σ. Although the proof of Step 2 was inspired by the arguments in [D] and [L], these arguments are greatly simplified by using the theory of complexes of groups developed by Haefliger in [H1] and [H2]. For an arbitrary Coxeter system (W, S) with associated Artin group A, we show that the subgroups A_T of A, generated by $T \in \mathcal{S}^f$, form a complex of groups. This complex of groups is developable, that is, it arises from the action of a group on a simplicial complex, explicitly, from the action of A on the modified Deligne complex Φ. We prove that the quotient manifold M is homotopy equivalent to the classifying space of this complex of groups and that Φ is homotopy equivalent to the universal covering space of this classifying space. The equivalence of Conjectures 1 and 2 follows.

We turn now to Step 3. Our program for proving the contractibility of Φ was motivated by the proofs of Gromov [G] and Moussong [M] that Σ can be given a piecewise Euclidean $CAT(0)$ metric. The complex Σ may be viewed as a quotient of $W \times K$, with copies of K glued along certain faces. K, and hence Σ, has a natural decomposition into combinatorial cubes. If we assign each of these cubes the metric of a regular Euclidean cube (as Gromov suggests), then the condition that links of vertices are $CAT(1)$ reduces to a combinatorial condition, namely that the links are "flag complexes" (see §4.2). Analyzing the links of vertices in Σ, we find that the links are joins of two types of complexes:

(i) the spherical Coxeter complex $\hat{\Sigma}_T$ associated to some finite W_T,

(ii) the geometric realization θ_T of the abstract simplicial complex

$$\mathscr{S}^f_{>T} = \{T' \in \mathscr{S}^f | T \subsetneqq T'\},$$

where $T \in \mathscr{S}^f$. It is well-known (cf. [Br, p. 29]) that the first of these is always a flag complex. The second is a flag complex for all $T \in \mathscr{S}^f$ if and only if (W, S) satisfies the following condition.

(FC): If $T \subset S$ and every pair of elements in T generates a finite subgroup of W, then T generates a finite subgroup of W.

Thus, Gromov's piecewise Euclidean structure on Σ (using regular cubes) is $CAT(0)$ if and only if (W, S) satisfies (FC).

The situation for Φ is similar; Φ may be viewed as a quotient of $A \times K$ and, as such, has a decomposition into combinatorial cubes which we can view as regular Euclidean cubes. In this case, the link of a vertex is a join of a spherical Deligne complex $\hat{\Phi}_T$ and the complex θ_T. We prove that every spherical Deligne complex is a flag complex. We conclude that, as for Σ, this metric on Φ is $CAT(0)$ if and only if (W, S) satisfies (FC). This gives our main theorem.

Theorem A. *Conjecture* 1 *holds for any Coxeter system* (W, S) *satisfying* (FC).

If W_T is a minimal example for which (FC) fails, then its fundamental chamber must be a simplex of dimension at least 2. It is well-known (cf. [Bo] or [La]) that such a Coxeter group is either an irreducible Euclidean reflection group (= an "affine Weyl group") or a hyperbolic reflection group. Moreover, in the hyperbolic case the dimension must be ≤ 4. (In dimension 2 these are the hyperbolic triangle groups; there are nine more examples in dimension 3 and five more in dimension 4.) Thus, condition (FC) holds for (W, S) whenever its Coxeter diagram does not contain a subdiagram of one of the above types. These forbidden subdiagrams are listed in Tables 1 and 2 in §4.3.

In his Ph.D. thesis [M], G. Moussong considered a different piecewise Euclidean metric on Σ. He identifies each combinatorial cube in K with a certain Euclidean convex polytope, the dihedral angles of which depend on W. These angles are defined in such a way that the induced metric on each spherical Coxeter complex $\hat{\Sigma}_T$ appearing in the link of a vertex is the natural metric, that is, the standard round metric on the sphere. Moussong proves that the induced piecewise spherical metrics on the complexes θ_T appearing in the links are $CAT(1)$ for all T. Since joins of $CAT(1)$ spaces are $CAT(1)$, it follows that Moussong's metric on Σ is $CAT(0)$ for all (W, S).

If we put the analoguous metric on Φ, the complexes θ_T in the links are again $CAT(1)$ by Moussong's result, and every apartment in $\hat{\Phi}_T$ is a round sphere, but it does not follow a priori that $\hat{\Phi}_T$ is $CAT(1)$. Hence, Moussong's metric on Φ is $CAT(0)$ if and only if the following conjecture holds.

Conjecture 3. *For each finite Coxeter system the natural metric on the associated Deligne complex* $\hat{\Phi}$ *(making each apartment into a round sphere) is* $CAT(1)$.

Thus, Conjecture 3 implies Conjecture 1. There is one case when Conjecture 3 is easy to prove: when the Coxeter group is a finite dihedral group. In this case Conjecture 3 follows from a lemma of Appel and Schupp [AS]. This is enough to prove Conjecture 1 whenever K is two-dimensional; that is, when any three distinct elements of S generate an infinite subgroup of W. This gives the following.

Theorem B. *Conjecture* 1 *holds for any Coxeter system such that* K *is two-dimensional.*

For example, Theorem B applies to Euclidean and hyperbolic triangle groups. It also applies in the case when, in the language of [AS], (W, S) is of "large type". In this case, Theorem B was proved by Hendriks [He].

In a subsequent paper, [CD3], we describe a finite CW-complex which is homotopy equivalent to M. This complex has the same dimension as K. In the situations where Conjecture 1 holds, this allows us to deduce quite a bit about the Artin group A. For example, its cohomological dimension is equal to $\dim K$.

1. The complex of Artin groups

In this section we establish some preliminaries on complexes of groups in general and complexes of Coxeter groups and Artin groups in particular.

(1.1) **Coxeter groups and Artin groups.** A *Coxeter matrix* is a symmetric matrix $M = (m_{ij})$ with $m_{ii} = 1$ for all i and $m_{ij} \in \{2, 3, \cdots, \infty\}$ for $i \neq j$. Associated to an $n \times n$ Coxeter matrix is a *Coxeter system* (W, S), where $S = \{s_1, \cdots, s_n\}$ is a finite set and W is the group with presentation

$$W = \langle s_1, \cdots, s_n | (s_i s_j)^{m_{ij}} = 1 \rangle$$

(with the convention that the relation is ignored if $m_{ij} = \infty$). If $T \subset S$, then let (W_T, T) be the Coxeter system defined by the corresponding $T \times T$ submatrix of M and let $\overline{W_T}$ be the subgroup of W generated by T.

Lemma 1.1.1 ([Bo, Ch. IV, §1.8, Théorème 2 (i)]). *The natural map* $W_T \to \overline{W_T}$ *is an isomorphism.*

Hence, we can view W_T as a subgroup of W. The groups W_T are called the *special subgroups* of W and their cosets wW_T are the *special cosets.* (By convention, $W_\varnothing = \{1\}$ and $W_\varnothing$ is also a special subgroup.)

Also associated to the Coxeter matrix (m_{ij}) is an *Artin group* (or "generalized braid group")

$$A = \langle s_1, \cdots, s_n | \mathrm{prod}(s_i, s_j; m_{ij}) = \mathrm{prod}(s_j, s_i; m_{ij}) \rangle,$$

where

$$\mathrm{prod}(s_i, s_j; m_{ij}) = \underbrace{s_i s_j s_i s_j \cdots}_{m_{ij}\text{-terms}}.$$

If the corresponding Coxeter group W is finite, we say A is of *finite type*. If we add to the presentation for A the relations $s_i^2 = 1$, then we get the presentation for W; hence, W is a quotient of A. Denote the quotient homomorphism by $q : A \to W$.

For $T \subset S$, let A_T denote the Artin group associated to (W_T, T) and let $\overline{A_T}$ be the subgroup of A generated by T.

Lemma 1.1.2. *The natural map $A_T \to \overline{A_T}$ is an isomorphism.*

This was proved by Deligne [D, Théorème 4.14(iii)] for A of finite type and by van der Lek [L, Theorem 4.13] for general A. We include our own proof in the case that A_T is of finite type in §3, as Corollary 3.25. Hence, we can identify A_T with its image in A. These subgroups are called the *special subgroups* of A.

The quotient homomorphism $q : A \to W$ has a set-theoretic section $m : W \to A$ defined as follows. For $w \in W$, let $s_1 \cdots s_k$, $s_i \in S$, be a word of minimal length representing w in the free group $F(S)$. Define $m(w)$ to be the image of $s_1 \cdots s_k$ in A. It follows from Tits' solution to the word problem for Coxeter groups in [T] that m is well-defined. The mapping m is not, of course, a homomorphism, but it does preserve special cosets.

Lemma 1.1.3. *If $w_1 W_T = w_2 W_T$, then $m(w_1)A_T = m(w_2)A_T$.*

Proof. It follows from [Bo, Ch. IV , Ex. 3, p. 37] that a special coset wW_T has a unique element w_0 of minimal length and that for any $w \in w_0 W_T$ we can write $w = w_0 u$ with $u \in W_T$ and $\text{length}(w) = \text{length}(w_0) + \text{length}(u)$. It then follows from the definition of m that $m(w) = m(w_0)m(u)$. A minimal word representing u will involve only the generators in T [Bo, Ch. IV, §1.8], hence $m(u) \in A_T$ and $m(w)A_T = m(w_0)A_T$. □

Given a Coxeter system (W, S), we define the *Coxeter complex* $\hat{\Sigma}$ as follows. For $n = \text{Card}(S) - 1$, let Δ_S be an n-simplex with vertices labeled by the elements of S. Denote by σ_T the face of Δ_S spanned by the vertices $\{s_i \mid s_i \notin T\}$. Define

$$\hat{\Sigma} = W \times \Delta_S / \sim,$$

where $(w_1, x_1) \sim (w_2, x_2)$ provided $x_1 = x_2$ and if σ_T is the (open) simplex containing x_1, then $w_1^{-1} w_2 \in W_T$.

There is an analogous complex for A, first introduced in [D, §2], which we call the *Deligne complex* and denote by $\hat{\Phi}$. Namely,

$$\hat{\Phi} = A \times \Delta_S / \sim,$$

where $(a_1, x_1) \sim (a_2, x_2)$ provided $x_1 = x_2$ and if σ_T is the (open) simplex containing x_1, then $a_1^{-1} a_2 \in A_T$.

The natural map $\Delta_S = 1 \times \Delta_S \to \hat{\Sigma}$ is an injection. Its image is called the *fundamental chamber* of $\hat{\Sigma}$; its translates under W are *chambers* of $\hat{\Sigma}$. Likewise for $\Delta_S = 1 \times \Delta_S \to \hat{\Phi}$.

The quotient map $q : A \to W$ induces a simplicial map $\hat{q} : \hat{\Phi} \to \hat{\Sigma}$. It follows from Lemma 1.1.3 that the (set-theoretic) section $m : W \to A$ of

q induces a simplicial section $\hat{m} : \hat{\Sigma} \to \hat{\Phi}$ of $\hat{q}$. The image of $\hat{m}$ in $\hat{\Phi}$ is called the *fundamental apartment* of $\hat{\Phi}$; its translates under A are called *apartments*. Since $\hat{q}$ is equivariant (with respect to q), it maps each apartment isomorphically onto $\hat{\Sigma}$.

Remark. The terminology of "chambers" and "apartments" is borrowed from the theory of buildings. In some respects $\hat{\Phi}$ is "building-like", but it fails to satisfy the axiom for buildings which requires that any two chambers belong to a common apartment.

(1.2) **Posets.** Let $\mathscr{P}$ be a partially ordered set (or *poset*). Associated to $\mathscr{P}$ is the *derived poset* $\mathscr{P}'$ whose elements are totally ordered chains $\sigma = (a_0 < a_1 < \cdots < a_k)$ in $\mathscr{P}$ with $\tau < \sigma$ if τ is a subchain of σ. The derived poset may be viewed as an abstract simplicial complex. We denote its geometric realization by $|\mathscr{P}'|$.

For an element $a \in \mathscr{P}$, define subposets

$$\mathscr{P}_{<a} = \{b \in \mathscr{P} | b < a\}$$

and, similarly, $\mathscr{P}_{\leq a}$, $\mathscr{P}_{>a}$ and $\mathscr{P}_{\geq a}$. Let $K = |\mathscr{P}'|$. The subcomplex

$$F_a = |(\mathscr{P}_{\geq a})'|$$

of K is called the *face* associated to a. The *open face* associated to a is the relative interior of F_a,

$$\mathring{F}_a = |(\mathscr{P}_{\geq a})'| - |(\mathscr{P}_{>a})'|.$$

The open faces partition K; that is, every point of K belongs to a unique open face.

Similarly, we define the *dual face* and the *open dual face* associated to a by

$$F_a^* = |(\mathscr{P}_{\leq a})'|, \qquad \mathring{F}_a^* = |(\mathscr{P}_{\leq a})'| - |(\mathscr{P}_{<a})'|.$$

Note that if $\mathscr{P}^{op}$ denotes the poset obtained by reversing the ordering on $\mathscr{P}$, then $\mathscr{P}'$ and $(\mathscr{P}^{op})'$ are isomorphic posets and the faces of $|\mathscr{P}'|$ are the dual faces of $|(\mathscr{P}^{op})'|$ and vice versa.

Example. Let X be a cell complex and let $\mathscr{P}$ be the set of cells of X ordered by reverse inclusion. Then $|\mathscr{P}'|$ can be naturally identified with the barycentric subdivision of X. The face F_c associated to a cell $c \subset X$ is then identified with c, and the dual face F_c^* is identified with the cone on the link of c in X.

(1.3) **Complexes of groups.** The notion of a "complex of groups" was introduced by A. Haefliger in [H1], [H2]. (Also, see [Co] and [S].) We review some basic material here and refer the reader to [H1], [H2] for more details.

Definition. Suppose $\mathscr{P}$ is a poset and X is the geometric realization of $\mathscr{P}'$. A *simple complex of groups* $G(X)$ on X is given by the following data:

(1) for each $a \in \mathscr{P}$, a group G_a,

(2) for each $a, b \in \mathscr{P}$ with $b < a$ an injective homomorphism $\psi_{ab} : G_b \to G_a$ such that if $c < b < a$, then $\psi_{ac} = \psi_{ab} \circ \psi_{bc}$.

Remark. Haefliger's definition of a "complex of groups" in [H2] is more general than the above in two ways. First of all, in place of the poset $\mathscr{P}$, he allows a small category without loop. Secondly, the homomorphism ψ_{ac} is required to agree with the composition $\psi_{ab} \circ \psi_{bc}$ only up to conjugation by an element of G_a (and this element must be kept track of). This more general definition plays no role in our situation, hence the term *simple* complex of groups.

Associated to a complex of groups $G(X)$ there is a classifying space $BG(X)$. In the case of a simple complex of groups it can be defined as follows. Let $CG(X)$ be the category with objects the elements of $\mathscr{P}$ and with

$$\mathrm{Hom}(b, a) = \begin{cases} G_a, & \text{if } b \le a, \\ \varnothing, & \text{otherwise.} \end{cases}$$

For $c < b < a$, the composition of $g \in \mathrm{Hom}(c, b)$ with $h \in \mathrm{Hom}(b, a)$ is defined to be $h\psi_{ab}(g) \in G_a = \mathrm{Hom}(c, a)$. The *classifying space* $BG(X)$ is the geometric realization of the nerve of the category $CG(X)$. There is an obvious functor from $CG(X)$ to $\mathscr{P}$ (viewed as a category) which gives rise to a projection $p : BG(X) \to X$.

The *fundamental group*, $\pi_1(G(X))$, is the fundamental group of the space $BG(X)$.

An alternate description of $\pi_1(G(X))$ is given in [H1, §3] with generators and relations being given in terms of the groups G_a and the edges of X. From this description, one can immediately deduce the following.

Lemma 1.3.1. *If $G(X)$ is a simple complex of groups and X is simply connected, then $\pi_1(G(X))$ is the colimit of the groups G_a with respect to the homomorphisms ψ_{ab}. In other words, if $\langle S_a | R_a \rangle$ is a presentation for G_a, then*

$$\left\langle \bigcup_a S_a \,\middle|\, \bigcup_a R_a,\ \psi_{ab}(s) = s,\ \forall b < a,\ s \in S_b \right\rangle$$

is a presentation for $\pi_1(G(X))$.

Suppose $\tilde{\mathscr{P}}$ is a poset and $\tilde{X}$ is the geometric realization of its derived complex. (Then $\tilde{X}$ is an "ordered" simplicial complex in the sense that the vertex set of each of its simplices is canonically ordered.) An order-preserving G-action on $\tilde{\mathscr{P}}$ is *without inversions* if, given $\tilde{a}, \tilde{b} \in \tilde{\mathscr{P}}$, with $\tilde{b} < \tilde{a}$, the isotropy subgroups $G_{\tilde{b}}$ and $G_{\tilde{a}}$ of $\tilde{b}$ and $\tilde{a}$ satisfy $G_{\tilde{b}} \subset G_{\tilde{a}}$.

Suppose now that G acts without inversions on $\tilde{\mathscr{P}}$. There is an induced simplicial G-action on $\tilde{X}$. Set $\mathscr{P} = \tilde{\mathscr{P}}/G$ and $X = \tilde{X}/G$. Then $\mathscr{P}$ is a "small category without loop". For simplicity, assume that $\mathscr{P}$ is a poset, i.e. there is at most one morphism between any two objects of $\mathscr{P}$. (This holds if we strengthen the "action without inversion" condition to read: if $\tilde{a} < \tilde{b}$ and $g \cdot \tilde{a} < \tilde{b}$, then $g \in G_{\tilde{b}}$.) Then X is the geometric realization of $\mathscr{P}'$, and we can associate a complex of groups $G(X)$ to this situation as follows. Choose a representative $\tilde{a}$ for each G-orbit $a \in \mathscr{P}$. The group associated to a is the isotropy subgroup $G_{\tilde{a}}$. For the moment fix $b \in \mathscr{P}$. If $b < a$, then, since the action is without inversions, there is a unique element $\tilde{a}'$ in the orbit a such that $\tilde{b} < \tilde{a}'$. Hence, there is an element $g_{ab} \in G$ such that $g_{ab}\tilde{a}' = \tilde{a}$. Define $\psi_{ab} : G_{\tilde{b}} \to G_{\tilde{a}}$ to be the composition

$$G_{\tilde{b}} \subset G_{\tilde{a}'} \xrightarrow{\mathrm{Ad}(g_{ab})} G_{\tilde{a}}$$

This gives the data for a complex of groups $G(X)$. (The definition in [H2, p. 281] is designed so that this will be true.) A complex of groups arising in this fashion is called *developable.*

In general, of course, $G(X)$ will not be simple. However, if $\mathscr{P}$ has a smallest element (as will be the case in our application), then the choices above can be modified so that each $g_{ab} = 1$, i.e., so that $G(X)$ is simple (cf. Proposition 2.2.1 in [H2]).

Theorem 1.3.2 (Haefliger [H1]). *A complex of groups $G(X)$ is developable if and only if the natural maps $G_a \to \pi_1(G(X))$ are injective for all $a \in \mathscr{P}$.*

Suppose that G, $\tilde{\mathscr{P}}$, $\tilde{X}$ and $G(X)$ are as above. Then the classifying space of $G(X)$ can be identified with the Borel construction

$$BG(X) = \tilde{X} \times_G EG$$

where EG is the universal covering space of BG $(= K(G, 1))$ (cf. [H2, Proposition 3.2.3]). It follows that there is a short exact sequence

$$1 \to \pi_1(\tilde{X}) \to \pi_1(G(X)) \to G \to 1.$$

Thus, $\tilde{X}$ is simply connected if and only if $\pi_1(G(X)) = G$. Moreover, if this is the case, then the universal cover of $BG(X)$ is homotopy equivalent to $\tilde{X}$.

(1.4) **Complexes of spaces.** In [H2], Haefliger also introduces the notion of a complex of spaces on an ordered simplicial cell complex X. We again restrict our attention to the case in which $X = |\mathscr{P}'|$ for a poset $\mathscr{P}$.

Definition. A *simple complex of spaces* over X is a topological space Y together with a continuous projection $p : Y \to X$ and a section $s : X^{(1)} \to Y$ of p over the 1-skeleton of X satisfying the following conditions.

(i) For each vertex a of X, $p^{-1}(a)$ is connected and there is a retraction $r_a : p^{-1}(F_a^*) \to p^{-1}(a)$ which is homotopic to the identity relative to $p^{-1}(a)$. Furthermore, $r_a \circ s$ must be constant on $X^{(1)} \cap F_a^*$.

(ii) Let $G_a = \pi_1(p^{-1}(a), s(a))$. For $b < a$, let $\psi_{ab} : G_b \to G_a$ be the map of fundamental groups induced by the composition $p^{-1}(b) \hookrightarrow p^{-1}(F_a^*) \xrightarrow{r_a} p^{-1}(a)$. Then ψ_{ab} is injective.

The data $\{G_a, \psi_{ab}\}$ gives rise to a simple complex of groups. (In the general definition of a "complex of spaces" in [H2], as before, one relaxes the condition on $\mathscr{P}$ to be a small category without loop, and in (i) it is only required that $r_a \circ s$ be constant on the vertices of F_a^*.)

Definition. If $p : Y \to X$ is a complex of spaces with associated complex of groups $G(X)$, we say that Y is a *topological realization* of $G(X)$. If Y is a

CW-complex and p is cellular, then Y is a *cellular realization* of $G(X)$. If $p^{-1}(a)$ is aspherical for every vertex a, then Y is an *aspherical realization* of $G(X)$.

An example of an aspherical cellular realization of $G(X)$ is the classifying space $BG(X)$.

Theorem 1.4.1 ([H2, Theorem 3.5.2]). *Every aspherical cellular realization of $G(X)$ is homotopy equivalent to $BG(X)$.*

Remark. It follows from van Kampen's theorem that if Y is a cellular realization of $G(X)$, then $\pi_1(Y) = \pi_1(G(X))$ (cf. [Co, Theorem 2.4]).

(1.5) **Complexes of Coxeter groups and Artin groups.** Suppose (W, S) is a Coxeter system. Let $\mathcal{S}$ be the poset of subsets of S ordered by inclusion, let $\mathcal{S}^f$ be the subposet defined by

$$\mathcal{S}^f = \{T \subseteq S \mid W_T \text{ is finite}\},$$

and let $K = |(\mathcal{S}^f)'|$. Let G stand for W or A, where A is the Artin group associated to (W, S). It follows from Lemmas 1.1.1 and 1.1.2 that the groups G_T and natural inclusions $G_T \to G_{T'}$ for $T < T'$ define a simple complex of groups $G(K)$ and that this complex of groups is developable. Since $\mathcal{S}^f$ has an initial object, $\emptyset$, K is simply connected, so it follows from Lemma 1.3.1 that $\pi_1(G(X)) = G$.

Remarks. (i) We could have considered the analogous complex of groups over the full poset $\mathcal{S}$. In later sections, however, it will be important that the vertex groups G_T are all finite (for $G = W$) or of finite type (for $G = A$).

(ii) In his Ph.D. thesis [S], B. Spieler showed that any complex of groups which supports a metric of "nonpositive curvature" (i.e., is locally $CAT(0)$) is developable. Moussong [M] showed that $W(K)$ always supports such a metric. In §4 we discuss the problem of assigning nonpositively curved metrics to $A(K)$.

Since $G(K)$ is developable, it arises from the action of G on a poset $\tilde{\mathcal{P}}$. The geometric realizations of the posets $\tilde{\mathcal{P}}$ for $G = W$ and $G = A$ will play a major role in the sections which follow, so we now describe them explicitly.

For $G = W$ or A, let

$$G\mathcal{S}^f = \{gG_T | g \in G, T \in \mathcal{S}^f\},$$

ordered by inclusion. There is an obvious order-preserving left action of G on $G\mathcal{S}^f$. The stabilizer of $gG_T \in G\mathcal{S}^f$ is the subgroup gG_Tg^{-1}; hence, G acts without inversion on $G\mathcal{S}^f$. The inclusion of $\mathcal{S}^f$ into $G\mathcal{S}^f$ as the cosets eG_T of the identity element e gives a splitting of the projection $G\mathcal{S}^f \to G\mathcal{S}^f/G = \mathcal{S}^f$. Using this splitting to choose representives for $\mathcal{S}^f$ in $G\mathcal{S}^f$, the complex of groups associated to the action of G on $G\mathcal{S}^f$ is precisely the complex $G(K)$ described above.

Let $\tilde{K}_G$ be the geometric realization of the derived complex of $G\mathcal{S}^f$. We have already observed that $\pi_1(G(K)) = G$. Thus, $BG(K) = \tilde{K}_G \times_G EG$ and its universal covering space is given by $\widetilde{BG}(K) = \tilde{K}_G \times EG$, which is homotopy equivalent to $\tilde{K}_G$.

Henceforth, we will use the following notation,

$$\Sigma = \tilde{K}_W = |(W\mathcal{S}^f)'|, \qquad \Phi = \tilde{K}_A = |(A\mathcal{S}^f)'|.$$

The complex Σ will be called the *modified Coxeter complex* and Φ the *modified Deligne complex*. The discussion above is summarized in the theorem below.

Theorem 1.5.1. (i) *The complex of Coxeter groups $W(K)$ is developable and its fundamental group is W. Its classifying space $BW(K)$ is*

$$\Sigma \times_W EW,$$

and hence, the universal covering space of $BW(K)$ is homotopy equivalent to the modified Coxeter complex Σ.

(ii) *The complex of Artin groups $A(K)$ is developable and its fundamental group is A. Its classifying space $BA(K)$ is*

$$\Phi \times_A EA,$$

and hence, the universal covering space of $BA(K)$ is homotopy equivalent to the modified Deligne complex Φ.

As suggested by the terminology, the complexes Σ and Φ are closely related to the ordinary Coxeter and Deligne complexes $\hat{\Sigma}$ and $\hat{\Phi}$, at least in the case when W is finite. To see this, note that the inclusion $\mathcal{S}^f \hookrightarrow W\mathcal{S}^f$ corresponds to an inclusion $K \hookrightarrow \Sigma$. The W-translates of K cover Σ, so Σ is a quotient of $W \times K$, namely

$$\Sigma = W \times K/\sim,$$

where $(w_1, x_1) \sim (w_2, x_2)$ provided $x_1 = x_2$ in K and if $\mathring{F}_T$ is the open face containing x_1, then $w_1^{-1}w_2 \in W_T$. Similarly, we have $K \hookrightarrow \Phi$ and

$$\Phi = A \times K/\sim,$$

where $(a_1, x_1) \sim (a_2, x_2)$ provided $x_1 = x_2$ in K and if $\mathring{F}_T$ is the open face containing x_1, then $a_1^{-1}a_2 \in A_T$. The translates of K in Σ (resp. Φ) are called *chambers* and K itself is called the *fundamental chamber* of Σ (resp. Φ).

Now recall the simplex Δ_S, with vertices labelled by S, used to define the Coxeter complex $\hat{\Sigma}$. With notation as in (1.1), the map $\sigma_T \mapsto T$ is an order reversing map from the poset $\mathrm{Simp}(\Delta_S)$ of simplices of Δ_S to $\mathcal{S}$. In fact, it is an isomorphism from $\mathrm{Simp}(\Delta_S)$ onto $(\mathcal{S}_{<S})^{op}$. It follows that $|\mathcal{S}'|$ $(= |(\mathcal{S}^{op})'|)$ is naturally isomorphic to the cone on the barycentric subdivision of Δ_S, where the cone point is the vertex v_S corresponding to $S \in \mathcal{S}$. For $T \neq S$ the face F_T of $|\mathcal{S}'|$ is identified under this isomorphism with the barycentric subdivision of the face σ_T of Δ_S. Now suppose that W is finite. Then $\mathcal{S} = \mathcal{S}^f$, so $K = |\mathcal{S}'|$. Thus, K is the cone on the barycentric subdivision of Δ_S. Moreover, since the translates of v_S $(= F_S)$ in Σ are all identified, we

conclude that for W finite, Σ is the cone on the barycentric subdivision of $\hat{\Sigma}$. Similarly, Φ is the cone on the barycentric subdivision of $\hat{\Phi}$.

As is the case of $\hat{\Sigma}$, $\hat{\Phi}$, the projection $q : A \to W$ induces a simplicial map $\overline{q} : \Phi \to \Sigma$ and the section $m : W \to A$ induces a simplicial section $\overline{m} : \Sigma \to \Phi$ of $\overline{q}$ (Lemma 1.1.3). The image of $\overline{m}$ in Φ is called the *fundamental apartment* of Φ, and its translates by A are called *apartments*. Since $\overline{q}$ is equivariant (with respect to q), it maps each apartment isomorphically onto Σ.

2. Hyperplane complements

As in the Introduction, suppose (W, S) is a Coxeter system acting as a linear reflection group on a real vector space V and that $\overline{I}$ is the Tits cone. We identify $V \otimes \mathbb{C}$ with $V \times V$ and the domain Ω with $V \times I$.

Let R denote the set of elements of W which act as reflections on V. (This is precisely the set of conjugates of the elements in S.) For $r \in R$, let H_r denote the hyperplane of V fixed by r. We are interested in the hyperplane complement

$$\mathcal{Y} = (V \times I) - \bigcup_{r \in R} (H_r \times H_r).$$

W acts freely on $\mathcal{Y}$, and we denote the quotient manifold by $M = \mathcal{Y}/W$.

The goal of this section is to replace $\mathcal{Y}$ by a combinatorial object. For $r \in R$, let P_r denote the fixed set of r acting on the modified Coxeter complex Σ. P_r is called the *wall* associated to r. Consider the diagonal action of W on $\Sigma \times \Sigma$. The set of points with nontrivial isotropy groups is the union of the $P_r \times P_r$ for all walls P_r of Σ. Let

$$Y = (\Sigma \times \Sigma) - \bigcup_{r \in R} (P_r \times P_r)$$

and let $Q = Y/W$ be the orbit space. In this section we prove that Y is W-equivariantly homotopy equivalent to $\mathcal{Y}$, and hence that Q is homotopy equivalent to M.

(2.1) **Reduction to $I \times I$.** For a group G and a G-space Z, denote the isotropy group at $z \in Z$ by G_z $(G_z = \{g \in G | gz = z\})$. An equivariant map $f : Z \to Z'$ is *isovariant* if for all $z \in Z$, $G_z = G_{f(z)}$. The map f is an *isovariant* (resp. *equivariant*) *homotopy equivalence* if there is an isovariant (resp. equivariant) map $h : Z' \to Z$ and isovariant (resp. equivariant) homotopies $f \circ h \sim \mathrm{id}_{Z'}$ and $h \circ f \sim \mathrm{id}_Z$.

We first reduce the study of the W-space $V \times I$ to that of $I \times I$, which can be more easily compared with $\Sigma \times \Sigma$.

Lemma 2.1.1. *The inclusion $i : I \times I \to V \times I$ is a W-isovariant homotopy equivalence.*

Proof. Define equivariant maps p' and p from $V \times I$ to I as follows. If $x \in I$, then set $p'(x, y) = x$. If $x \notin I$, then let $p'(x, y)$ be the point where the line segment from x to y intersects $\partial \overline{I}$ (see Figure 1). Let $p(x, y) = \frac{1}{2}p'(x, y) + \frac{1}{2}y$. Then $p(x, y)$ is on the line segment from x to y and lies in I.

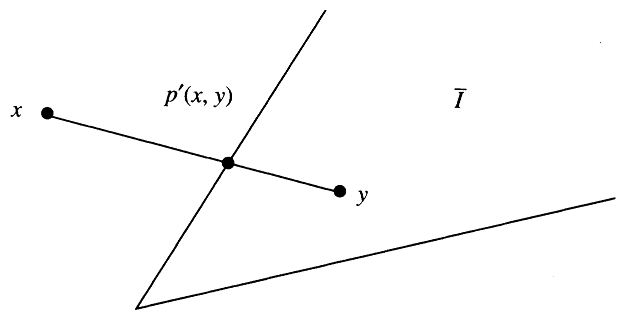

FIGURE 1

Define $r_1 : V \times I \to I \times I$ by $r_1(x, y) = (p(x, y), y)$. Then r_1 is an equivariant deformation retraction. Indeed, the straight line homotopy

$$r_t(x, y) = (tp(x, y) + (1-t)x, y)$$

is the required deformation of $i \circ r_1$ to $\mathrm{id}_{V \times I}$.

It remains to show that r_t is isovariant. Note that if $w \in W$ fixes two distinct points in V, then it fixes the affine line which they span. Suppose $w \in W$ fixes $r_t(x, y)$. Then it fixes y and the point $tp(x, y) + (1-t)x$ on the line from x to y; hence, it also fixes x. Thus, w fixes (x, y), which shows that r_t is isovariant. □

(2.2) **Mirror structures.** A *mirror structure* on a space X indexed by a set S is a collection of closed subspaces $\{X_s\}_{s \in S}$. Given a mirror structure and a subset T of S, define $X_T = \bigcap_{s \in T} X_s$ if T is nonempty, and $X_\varnothing = X$. The subspaces X_T are the *faces* of X. For $x \in X$, set $S(x) = \{s \in S | x \in X_s\}$, and define

$$\mathring{X}_T = \{x \in X_T | S(x) = T\}, \qquad \partial X_T = X_T - \mathring{X}_T.$$

If X has a mirror structure indexed by S and $T \subseteq S$, then it has an induced mirror structure indexed by T (by forgetting the mirrors indexed by $S - T$).

Suppose that X' has a mirror structure indexed by S and X has a mirror structure indexed by T, for $T \subseteq S$. A map $f : X \to X'$ is *T-face-preserving* if $T(x) = T(f(x))$ for all $x \in X$. If $T = S$, then we shall simply say that f is *face-preserving.*

A mirror structure is *collared* if for every subset T of S, there exists a map $c_T : \partial X_T \times [0, 1] \to X_T$ such that c_T is a homeomorphism onto a closed neighborhood of ∂X_T in X_T and such that the restriction of c_T to $\partial X_T \times 0$ is the inclusion.

Let $\mathcal{T}$ be a collection of subsets of S. A mirror structure is *defined over* $\mathcal{T}$ if $X_T = \varnothing$ for $T \notin \mathcal{T}$. It is *contractible over* $\mathcal{T}$ if it is defined over $\mathcal{T}$ and if $\mathring{X}_T$ is contractible for $T \in \mathcal{T}$.

Example 2.2.1. (i) In the notation of §1.5, let $X = K$ $(= |(\mathcal{S}^f)'|)$ and define a mirror structure by $X_s = F_{\{s\}}$, where F_T denotes the face $|(\mathcal{S}^f_{\geq T})'|$ as in §1.3. This mirror structure is defined over $\mathcal{S}^f$, and $X_T = F_T$ for $T \in \mathcal{S}^f$. Since X_T is a cone over ∂X_T, it is collared and contractible over $\mathcal{S}^f$.

(ii) In the notation of Vinberg's theorem (stated in the Introduction), let $X = C^f$ and consider the mirror structure given by $X_s = C^f \cap H_s$, which is defined over $\mathcal{S}^f$. Since $\overset{\circ}{X}_T$ is a convex subset of V, the mirror structure is contractible over $\mathcal{S}^f$. Since X_T is a smooth manifold with corners and its boundary is ∂X_T, it follows from the collared neighborhood theorem that this mirror structure is collared.

A face-preserving map $f : X \to X'$ is a *face-preserving homotopy equivalence* if there is a face-preserving map $g : X' \to X$ such that both composites are homotopic to the identity maps via face-preserving homotopies.

Lemma 2.2.2. *Let X, X' be spaces with mirror structures indexed by S, and let $\mathcal{T}$ be a collection of subsets of S.*

(i) *Suppose the mirror structure of X is collared and defined over $\mathcal{T}$ and the mirror structure on X' is contractible over $\mathcal{T}$. Then there is a face-preserving map $f : X \to X'$, and it is unique up to face-preserving homotopy.*

(ii) *If the mirror structures on X and X' are both collared and contractible over $\mathcal{T}$, then f is a face-preserving homotopy equivalence.*

Proof. The proof of (i) is a standard inductive argument. We may suppose by induction that f is defined on ∂X_T. Using the contractibility of $\overset{\circ}{X}{}'_T$, we then extend it to a map from X_T to X'_T which is constant on $X_T - c_T(\partial X_T \times [0, 1])$. Similarly, for a homototpy between two such maps. This proves (i). Statement (ii) follows easily from (i). □

Next we generalize the construction of $\hat{\Sigma}$ in §1.1. Given a Coxeter system (W, S) and a mirror structure $\{X_s\}_{s \in S}$; define a space

$$\mathcal{X}(W, X) = (W \times X)/\sim,$$

where $(w_1, x_1) \sim (w_2, x_2)$ if $x_1 = x_2$ and $w_1^{-1} w_2 \in W_{S(x_1)}$. For example, $\Sigma = \mathcal{X}(W, K)$.

Let $[w, x]$ denote the image of (w, x) in $\mathcal{X}(W, X)$. The group W acts on $\mathcal{X}(W, X)$ via the formula $w' \cdot [w, x] = [w'w, x]$. In the next lemma we collect some easy facts about this construction. (For the proofs of (i), (ii) and (iii), see Propositions 1, 2 and 4 of [V, pp. 1088-1089].)

Lemma 2.2.3 (Vinberg). (i) *The map $\mathcal{X}(W, X) \to X$ defined by $[w, x] \to x$ induces a homeomorphism $\mathcal{X}(W, X)/W \to X$.*

(ii) *The map $X \to \mathcal{X}(W, X)$ defined by $x \to [1, x]$ is a homeomorphism onto its image (which we identify with X).*

(iii) *If X is Hausdorff and the mirror structure is defined over $\mathcal{S}^f$, then the W-action is proper.*

(iv) *If Z is a W-space and $\varphi : X \to Z$ is a map such that $s(\varphi(x)) = \varphi(x)$ for all $s \in S$ and $x \in X_s$, then φ extends to a W-equivariant map $\mathcal{X}(W, X) \to Z$ by the formula $[w, x] \to w\varphi(x)$.*

(v) *If $f : X \to X'$ is a face-preserving map of spaces with mirror structures, then f extends to an isovariant map $\hat{f} : \mathcal{X}(W, X) \to \mathcal{X}(W, X')$ by $[w, x] \to [w, f(x)]$. If f is a face-preserving homotopy eqivalence, then $\hat{f}$ is an isovariant homotopy equivalence.*

Proposition 2.2.4. *There is an isovariant homotopy equivalence between* Σ *and* I.

Proof. Applying part (iv) of Lemma 2.2.3 to the inclusion $C^f \hookrightarrow I$, we see that $\mathscr{X}(W, C^f)$ can be identified with I. We also have $\Sigma = \mathscr{X}(W, K)$. As observed in Example 2.2.1, the mirror structures on K and C^f are collared and contractible over $\mathscr{S}^f$; hence, the proposition follows from Lemma 2.2.2. □

Corollary 2.2.5. $\Sigma \times \Sigma$ *is* W*-isovariantly homotopy equivalent to* $I \times I$, *and hence* Y *is* W*-equivariantly homotopy equivalent to the hyperplane complement* $\mathscr{Y}$.

Suppose X is a space with mirror structure. For each $s \in S$, let $H_{\{s\}}(X)$ be the *half-space* of $\mathscr{X}(W, X)$ containing X defined by

$$H_{\{s\}}(X) = \bigcup_{\ell(ws)>\ell(w)} wX$$

where the union is over all w in W with $\ell(ws) > \ell(w)$ (ℓ denotes the word length). For $T \subseteq S$, define

$$H_T(X) = \bigcap_{s\in T} H_{\{s\}}(X).$$

Then $H_T(X)$ has a mirror structure indexed by T defined by intersecting with those walls of $\mathscr{X}(W, X)$ which are indexed by T. Applying Lemma 2.2.3(iv) with $W = W_T$, $Z = \mathscr{X}(W, X)$, and $\varphi : H_T(X) \hookrightarrow \mathscr{X}(W, X)$ the inclusion, we obtain a homeomorphism $\mathscr{X}(W_T, H_T(X)) \to \mathscr{X}(W, X)$ which we regard as an identification.

For $T \in \mathscr{S}^f$, let K_T and Σ_T denote, respectively, the fundamental chamber and modified Coxeter complex for the finite Coxeter system (W_T, T). There is a natural identification of K_T with the subset $F_T^* \subset K$, where F_T^* is the dual face of T, discussed in §1.3.

Lemma 2.2.6. *The inclusion* $F_T^* \hookrightarrow K$ *induces a* W_T*-isovariant homotopy equivalence* $\Sigma_T \hookrightarrow \Sigma$.

Proof. We have

$$\Sigma_T = \mathscr{X}(W_T, F_T^*), \qquad \Sigma = \mathscr{X}(W_T, H_T(K)).$$

The mirror structures on F_T^* and $H_T(K)$ are both collared (clearly). By Lemmas 2.2.2(ii) and 2.2.3(v), it suffices to show these mirror structures are contractible over $\mathscr{T}$, the collection of all subsets of T. This is true for F_T^* by Example 2.2.1(i). To prove the mirror structure on $H_T(K)$ is contractible we use the homotopy equivalence $\Sigma \to I$ of Proposition 2.2.4. This restricts to a T-face-preserving homotopy equivalence $H_T(K) \to H_T(C^f)$, where $H_T(C^f)$ is an intersection of half-spaces of I. Since the faces of $H_T(C^f)$ are defined by intersecting with hyperplanes, they are convex and hence contractible. □

3. A COMPLEX OF SPACES

The projection of $\Sigma\times\Sigma$ onto the first factor restricts to a W-equivariant map $Y \to \Sigma$, and this induces a map of orbit spaces $p : Q \to K$ (where $Q = Y/W$ and $K = \Sigma/W$). We set up some notation. For each $T \in \mathcal{S}^f$, we denote by v_T the vertex of K corresponding to T, by F_T^* the dual face of T in K, by Y_T the nonsingular part of $\Sigma_T \times \Sigma_T$; and $Q_T = Y_T/W_T$.

(3.1) **A decomposition of** Q. Consider the decomposition of K into dual faces $\{F_T^*\}_{T\in\mathcal{S}^f}$ and its pullback to a decomposition $\{p^{-1}(F_T^*)\}$ of Q. The purpose of this subsection is to prove the following.

Proposition 3.1.1. *Suppose $T \in \mathcal{S}^f$. Then $p^{-1}(F_T^*)$ satisfies the following.*

(i) *$p^{-1}(F_T^*)$ is homotopy equivalent to Q_T. Hence, $p^{-1}(F_T^*)$ is aspherical with fundamental group A_T.*

(ii) *If $T' \subseteq T$, then the map of fundamental groups $A_{T'} \to A_T$ induced by the inclusion $p^{-1}(F_{T'}^*) \subseteq p^{-1}(F_T^*)$ is the natural homomorphism (up to conjugacy).*

(iii) *The inclusion $p^{-1}(F_T^*) \hookrightarrow Q$ induces an injection on fundamental groups.*

Proof. (i) Each component of the inverse image of F_T^* in Σ (under the orbit map $\Sigma \to K$) is isomorphic to Σ_T. Hence,

$$p^{-1}(F_T^*) = ((\Sigma_T \times \Sigma) \cap Y)/W_T.$$

By Lemma 2.2.6, the right hand side is homotopy equivalent to Y_T/W_T ($= Q_T$). By Corollary 2.2.5, Q_T is homotopy equivalent to $\mathcal{Y}_T/W_T$, where $\mathcal{Y}_T$ is the complex hyperplane complement corresponding to the orthogonal reflection group W_T on $\mathbb{R}^T$. By Deligne's theorem [D, Théorème 4.4], $\mathcal{Y}_T/W_T$ is aspherical with fundamental group A_T, which proves the second sentence of (i).

(ii) It is clear from Deligne's proof that for $T' \subset T$, any map $\mathcal{Y}_{T'}/W_{T'} \to \mathcal{Y}_T/W_T$ induced by a T'-face-preserving map of fundamental chambers induces the standard homomorphism $A_{T'} \to A_T$. Since $F_{T'}^* \hookrightarrow F_T^*$ is T'-face-preserving, we have a diagram

$$\begin{array}{ccc} p^{-1}(F_{T'}^*) & \xrightarrow{\sim} & \mathcal{Y}_{T'}/W_{T'} \\ \downarrow & & \downarrow \\ p^{-1}(F_T^*) & \xrightarrow{\sim} & \mathcal{Y}_T/W_T \end{array}$$

in which all arrows are induced by T'-face-preserving maps of fundamental chambers. By Lemma 2.2.2(i) such maps are unique up to homotopy; hence, the diagram commutes up to homotopy. So, (ii) holds.

(iii) Let

$$Z_T = (\Sigma_T \times \Sigma) \cap Y = (\Sigma_T \times \Sigma) - \bigcup_{r\in W_T} P_r \times P_r,$$

where the union is over all reflections r in W_T. In other words, Z_T is a component of the inverse image of $p^{-1}(F_T^*)$ in Y (so, $Z_T/W_T = p^{-1}(F_T^*)$).

By Lemma 2.2.6, the inclusion

$$Z_T \hookrightarrow \Sigma \times \Sigma - \bigcup_{r \in W_T} P_r \times P_r$$

is a W_T-equivariant homotopy equivalence. Since the inclusion factors through Y, the induced map $Z_T/W_T \hookrightarrow Y/W_T$ gives rise to an injection on fundamental groups. The same holds for $Z_T/W_T \hookrightarrow Y/W$, since $Y/W_T \to Y/W$ is the projection map of a covering space. □

(3.2) **A thickened version of Q.** We would like to say that Proposition 3.1.1 means that $p : Q \to K$ is a complex of spaces over K with $A(K)$ as its associated complex of groups. Unfortunately, this is not quite true. The problem is that, although the inverse images of the dual cells are correct, the inverse images of the vertices are not. To remedy this, we define $\hat{Q}$, a "thickened" version of Q.

Recall that the dual face F_T^* is the geometric realization of $(\mathcal{S}^f_{\leq T})'$ and that $\overset{\circ}{F}{}_T^* = |(\mathcal{S}^f_{\leq T})'| - |(\mathcal{S}^f_{<T})'|$. Each point x in K belongs to a unique "open" dual face $\overset{\circ}{F}{}_T^*$, for some $T \in \mathcal{S}^f$.

The space $\hat{Q}$ is the subspace of $K \times Q$ defined by

$$\hat{Q} = \{(x, q) \in K \times Q | x \in \overset{\circ}{F}{}_T^* \Rightarrow q \in p^{-1}(F_T^*)\}.$$

There is a projection map $\hat{p} : \hat{Q} \to K$ defined by $\hat{p}(x, q) = x$. Choose a basepoint $q_0 \in Q$ lying over the central vertex $v_\emptyset$ of K. Since $v_\emptyset$ belongs to each dual face, the point (x, q_0) lies in $\hat{Q}$ for all $x \in K$. Hence, the map $s : K \to \hat{Q}$ defined by $x \to (x, q_0)$ is a well-defined section of $\hat{p}$.

Theorem 3.2.1. *$\hat{p} : \hat{Q} \to K$ is an aspherical realization of $A(K)$.*

Proof. First we show that $\hat{p} : \hat{Q} \to K$ is a simple complex of spaces over K. (See §1.4.) Thus, we must define, for each $T \in \mathcal{S}^f$, a deformation retraction $r_T : \hat{p}^{-1}(F_T^*) \to \hat{p}^{-1}(v_T)$ such that $r_T \circ s$ is constant on the 1-skeleton of F_T^*. We have

$$\hat{p}^{-1}(v_T) = v_T \times p^{-1}(F_T^*), \qquad \hat{p}^{-1}(F_T^*) \subseteq F_T^* \times p^{-1}(F_T^*).$$

The map r_T defined by $r_T(x, q) = (v_T, q)$ is clearly a deformation retraction (since F_T^* is contractible to v_T) and on F_T^*, $r_T \circ s$ is the constant map $x \to (v_T, q_0)$. Thus, $\hat{Q}$ is a complex of spaces over K.

It follows from Proposition 3.1.1 that the associated complex of groups is $A(K)$ (since the fundamental group of $\hat{p}^{-1}(v_T) = p^{-1}(F_T^*)$ is A_T) and that it is an aspherical realization of $A(K)$ (since $p^{-1}(F_T^*)$ is aspherical). □

Corollary 3.2.2. (i) $\pi_1(\hat{Q}) = A$.

(ii) *$\hat{Q}$ is homotopy equivalent to $BA(K)$.*

Proof. Strictly speaking, to deduce this from Theorem 3.2.1 one should have that the complex of spaces $\hat{p} : \hat{Q} \to K$ is cellular; that is, that the fibers of $\hat{p}$

are CW-complexes. This will be true only after a slight modification of $\hat{Q}$. One replaces $\hat{Q}$ by a slightly smaller homotopy equivalent space which is a subcomplex of a fine subdivision of $K \times \Sigma \times \Sigma$. The details are left to the reader. □

There is a natural inclusion $i : Q \to \hat{Q}$ defined by $i(q) = (p(q), q)$ and a retraction $r : \hat{Q} \to Q$ given by $r(x, q) = q$.

Proposition 3.2.3. *Q is a deformation retract of $\hat{Q}$.*

Proof. We have $i \circ r(x, q) = (p(q), q)$, and the points x and $p(q)$ both belong to F_T^* if $x \in \overset{\circ}{F}{}_T^*$. We construct a homotopy from the identity to $i \circ r$ by deforming x to $p(q)$ in F_T^*. As we shall explain in detail in §4.3, the subcomplex F_T^* can be canonically identified with a regular Euclidean cube (of dimension $\mathrm{Card}(T)$) so that if $T' \subset T$, then $F_{T'}^*$ corresponds to a face of the cube for F_T^*. (F_T^* is simplicially isomorphic to the cone on the barycentric subdivision of the simplex on T, and this complex is isomorphic to a standard subdivision of the cube.) In any case, since F_T^* can be canonically identified with a convex subset of Euclidean space, for any $x_1, x_2 \in F_T^*$ and $t \in [0, 1]$, $tx_1 + (1-t)x_2$ is a well-defined point in F_T^*. Hence,

$$r_t(x, q) = (tp(q) + (1-t)s, q)$$

is a well-defined homotopy from the identity to $i \circ r$. □

Corollary 3.2.4. (i) $\pi_1(Q) = A$.
(ii) *Q is homotopy equivalent to $BA(K)$.*

Corollary 3.2.5 (van der Lek [L, Theorem 4.13]). *$A(K)$ is developable. (In other words, for each $T \in \mathcal{S}^f$, the standard homomorphism $A_T \to A$ is injective).*

Proof. This is immediate from Proposition 3.1.1(iii) and part (i) of the previous corollary. □

From Theorem 1.5.1(ii) and Corollary 3.2.4(ii), we get the following.

Corollary 3.2.6. *Q is aspherical if and only if Φ is contractible.*

By (2.2.5), $\mathcal{Y}/W$ is homotopy equivalent to Q (where $\mathcal{Y}$ denotes the hyperplane complement). Hence, the above corollary reduces the $K(\pi, 1)$-problem to the problem of showing that Φ is contractible.

Remark. The space $\hat{Q}$ is an artifact of our desire to frame the above results in the context of [H2]; it could have been dispensed with. An alternative argument would be to consider the universal covering space $\tilde{Y}$ of Y ($\tilde{Y}$ is also the universal covering of Q) and the open cover of $\tilde{Y}$ by the path components of the inverse images of open stars of vertices in K. Using the arguments in [D, §3] one can then show that each open set in this cover, as well as each nonempty intersection of such open sets, is contractible and that the nerve of this open cover is Φ. It follows that $\pi_1(Q) = A$, that $\Phi \times_A \tilde{Y}$ is homotopy equivalent to $BA(K)$, and hence, as before, that Q is aspherical if and only if Φ is contractible.

4. METRICS ON Φ

Gromov [G] and Moussong [M] defined piecewise Euclidean metrics on the modified Coxeter complex. In this section we discuss the analogous metrics for the modified Deligne complex Φ. We show that, under the hypotheses of Theorem A of the Introduction, Gromov's metric on Φ satisfies $CAT(0)$, and hence Φ is contractible. Conjecturally, Moussong's metric is always $CAT(0)$.

(4.1) **$CAT(r)$-metrics.** This terminology was introduced by Gromov in [G]. ("CAT" stands for "comparison of Alexandrov and Toponogov".) The idea is to define a notion for singular metric spaces which corresponds to "sectional curvature bounded above by r" for Riemannian manifolds. The theory was developed, in a context applicable to our situation, by Bridson in [B].

Let (X, d) be a metric space. A *geodesic segment* in X is the image of an isometric embedding $\gamma : [a, b] \to X$ of an interval into X. One says that (X, d) is a *geodesic space* (or a *length space*) if any two points in the same path component of X can be connected by a geodesic segment.

As r is positive, zero or negative, let $\mathbb{M}_r^n$ stand for, respectively, the n-sphere of radius $1/\sqrt{r}$, Euclidean n-space, or hyperbolic n-space of curvature r.

A geodesic space (X, d) has a *$CAT(r)$-metric* (or is a *$CAT(r)$-space*) if for any geodesic triangle in X, the distance from a vertex to a point on the opposite side is less than or equal to the distance between the corresponding points on a comparison triangle in $\mathbb{M}_r^2$.

If X is a connected $CAT(r)$-space with $r \leq 0$, then there is a *unique* geodesic segment connecting any two points in X, and such geodesic segments vary continuously with the endpoints. It follows that X is contractible.

An *$\mathbb{M}_r$-structure* on a cell complex X means an identification of each cell of X with a convex cell in $\mathbb{M}_r^n$, for some n, so that common faces are congruent. If $r = 0$ (resp., $r = 1$), then we call such a structure *piecewise Euclidean* (resp., *piecewise spherical*). If the number of isometry types of cells in the $\mathbb{M}_r$-structure is finite, the we say it satisfies the *finite shape condition* (FS).

An $\mathbb{M}_r$-structure defines an *intrinsic metric* on X, by taking $d(x, y)$ to be the infimum of the lengths of paths from x to y which are piecewise geodesic in the sense that they are geodesic in each cell. If X is locally finite or if it satifies (FS), then Moussong [M] and Bridson [B] have shown that this infimum is actually attained. Hence, (X, d) is a geodesic space. The question of whether d is a $CAT(r)$-metric is more subtle and is discussed in the next subsection.

(4.2) **Link conditions.** We assume that all our cell complexes are finite dimensional.

Given an $\mathbb{M}_r$-structure on X, it is possible to determine if the resulting intrinisc metric is $CAT(r)$ by looking at links of vertices in X.

For a vertex v of a convex cell σ in $\mathbb{M}_r^n$, define $Lk(v, \sigma)$ as the set of inward pointing unit tangent vectors at v; it is naturally a convex cell in a standard sphere. It follows that

$$Lk(v, X) = \bigcup_{v \in \sigma} Lk(v, \sigma)$$

is a piecewise spherical cell complex.

The following theorem was stated in [G, 4.2.A, p. 120] and proved under the (FS) hypothesis in [B, p. 396].

Theorem 4.2.1 (Gromov, Bridson). *Suppose that X is a cell complex with $\mathbb{M}_r$-structure, that $r \leq 0$, that X is simply connected, and that it satisfies* (FS). *Then X is a $CAT(r)$-space if and only if $Lk(v, X)$ is a $CAT(1)$-space for every vertex v in X.*

In general, proving that links are $CAT(1)$-spaces is no easy matter. However, there is one situation in which the problems can be reduced to a combinatorial one: when the cells of $Lk(v, X)$ are "all right" simplices.

Definition. An n-simplex σ in $\mathbb{S}^n$ is *all right* if each of its edge lengths is $\frac{\pi}{2}$. (In other words, σ is isometric to the intersection of $\mathbb{S}^n$ with the simplicial cone in $\mathbb{R}^{n+1}$ spanned by the standard basis.)

Remark. The link of a vertex in a regular Euclidean cube is an all right simplex. Hence, if X is piecewise Euclidean and the cells are regular cubes (of varying dimensions), then each cell of $Lk(v, X)$ is an all right simplex.

Definition. A simplicial complex L is a *flag complex* if whenever $\{v_0, \cdots, v_k\}$ is a set of distinct vertices which are pairwise joined by edges, then $\{v_0, \cdots, v_k\}$ spans a k-simplex in L. (In other words, L is "determined by its 1-skeleton".)

Proposition 4.2.2 (Gromov [G, p. 122]). *A piecewise spherical simplicial complex made up of all right simplices is a $CAT(1)$-space if and only if it is a flag complex.*

Corollary 4.2.3. *A simply connected, piecewise Euclidean complex made up of regular Euclidean cubes is a $CAT(0)$-space if and only if the link of each vertex is a flag complex.*

Remark. It is not necessary to assume, in Proposition 4.2.2, that the simplicial complex is finite or even locally finite. In fact, since a geodesic segment in an all right simplicial complex must lie in some finite subcomplex, we see that the general case of Proposition 4.2.2 follows from the special case where the complex is finite. A similar remark applies to Proposition 4.2.4, below.

Proposition 4.2.2 was generalized by G. Moussong [M] as follows.

Definition. A spherical simplex σ in $\mathbb{S}^n$ has *size* $\geq \frac{\pi}{2}$ if each of its edge lengths is at least $\frac{\pi}{2}$.

Definition. A piecewise spherical simplicial complex L is a *metric flag complex* if whenever $\{v_0, \cdots, v_k\}$ is a set of distinct vertices such that the v_i are pairwise joined by edges and such that there exists a k-simplex in $\mathbb{S}^k$ with the same edge lengths, then $\{v_0, \cdots, v_k\}$ spans a simplex in L. (In other words, L is "metrically determined by its 1-skeleton".)

Proposition 4.2.4 (Moussong [M]). *A piecewise spherical simplicial complex all of whose simplices have size $\geq \frac{\pi}{2}$ is a $CAT(1)$-space if and only if it is a metric flag complex.*

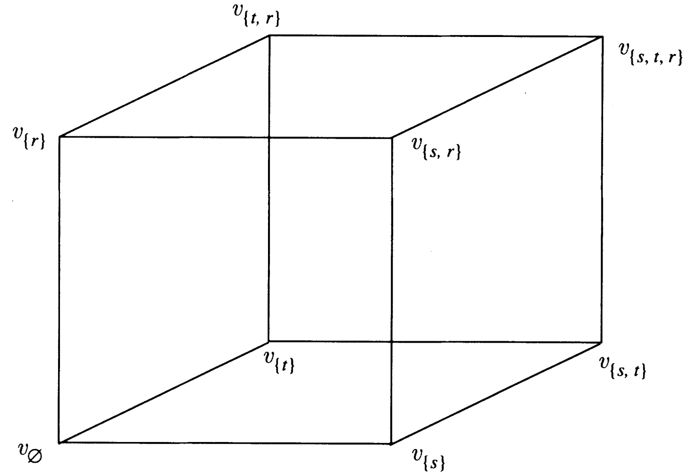

FIGURE 2. $F^*_{\{s,t,r\}}$ viewed as a cube

(4.3) **Cubical metrics on K, Σ and Φ.** As in §1, let (W, S) be a Coxeter system and K the geometric realization of the derived complex of $\mathcal{S}^f$. For each $T \in \mathcal{S}^f$, the dual face

$$F_T^* = |(\mathcal{S}^f_{\leq T})'|$$

is the cone on the barycentric subdivision of a simplex (with cone point $v_\emptyset$). This is isomorphic to a standard subdivision of a cube of dimension $\mathrm{Card}(T)$. The vertices of the cube are the vertices of K which lie in F_T^*, i.e., those of the form $v_{T'}$, $T' \subseteq T$. The cubical faces of F_T^* are the subsets of the form $F_{T_1} \cap F^*_{T_2}$ where $T_1 \subseteq T_2 \subseteq T$ (see Figure 2). Hence, the cubical faces of K which contain v_T are those of the form $F_{T_1} \cap F^*_{T_2}$, where $T_1 \subseteq T \subseteq T_2$ and $T_2 \in \mathcal{S}^f$.

Identify each such cube (and its translates) with a regular Euclidean cube of edge length 1. This defines a piecewise Euclidean, cubical structure on K, as well as on the modified Coxeter complex Σ and on the modified Deligne complex Φ. We wish to determine when the resulting intrinsic metrics on K, Σ and Φ are $CAT(0)$. By Corollary 4.2.3 this reduces to determining when links of vertices are flag complexes.

Let Δ_T be the simplex on T and let $\hat{\Sigma}_T$ and $\hat{\Phi}_T$ denote, respectively, the (unmodified) Coxeter complex and (unmodified) Deligne complex, i.e.,

$$\hat{\Sigma}_T = (W_T \times \Delta_T)/\sim, \qquad \hat{\Phi}_T = (A_T \times \Delta_T)/\sim.$$

Lemma 4.3.1. *Let $T \in \mathcal{S}^f$, and let v_T be the corresponding vertex of K. Then, with respect to the cubical structures discussed above, we have the following decomposition of links as joins:*

(i) $Lk(v_T, K) = \Delta_T * Lk(v_T, F_T)$,

(ii) $Lk(v_T, \Sigma) = \hat{\Sigma}_T * Lk(v_T, F_T)$,

(iii) $Lk(v_T, \Phi) = \hat{\Phi}_T * Lk(v_T, F_T)$.

Proof. As $X = K, \Sigma$ or Φ, let G_T stand for $\{1\}$, W_T or A_T and let X_T stand for F_T^*, Σ_T or Φ_T, respectively. The cubical faces of X containing v_T are the G_T-translates of the cubes $F_{T_1} \cap F_{T_2}^*$, where $T_1 \subseteq T \subseteq T_2$. Such a cube can be decomposed as a Cartesian product:

$$\begin{aligned} F_{T_1} \cap F_{T_2}^* &= (F_{T_1} \cap F_T^*) \times (F_T \cap F_{T_2}^*) \\ &\subseteq F_T^* \times F_T. \end{aligned}$$

Thus, the cubes containing v_T lie in $(G_T F_T^*) \times F_T$ (since G_T fixes F_T). Since $G_T F_T^*$ is naturally isometric to X_T it follows that

$$Lk(v_T, X) = Lk(v_T, X_T) * Lk(v_T, F_T).$$

Since $Lk(v_T, X_T)$ can be identified with $\Delta_T, \hat{\Sigma}_T$ or $\hat{\Phi}_T$ as $X_T = F_T^*, \Sigma_T$ or Φ_T, the result follows. □

The join $A * B$ of two simplicial complexes is a flag complex if and only if both A and B are flag complexes. Thus we need to analyze the terms on the right hand sides of the formulas in Lemma 4.3.1 separately. The simplex Δ_T is obviously a flag complex. So is $\hat{\Sigma}_T$. (This follows easily from its description as a poset of cells arising from a finite reflection group; cf. [Br, p. 29].) The following lemma is the main technical result needed to prove that Φ is contractible. Its proof occupies §4.5.

Lemma 4.3.2. *$\hat{\Phi}_T$ is a flag complex for all $T \in \mathcal{S}^f$.*

On the other hand, we shall see that $Lk(v_T, F_T)$ is not always a flag complex.

We recall that $\mathcal{S}^f_{>\emptyset}$ is an abstract simplicial complex: its vertex set is S, and a subset T of S is a simplex if and only if $T \in \mathcal{S}^f_{>\emptyset}$.

Lemma 4.3.3. (i) $Lk(v_\emptyset, F_\emptyset) = \mathcal{S}^f_{>\emptyset}$.

(ii) $Lk(v_T, F_T) = Lk(T, \mathcal{S}^f_{>\emptyset})$, *for* $T \in \mathcal{S}^f_{>\emptyset}$.

Proof. The cubes of F_T which properly contain v_T are intersections of the form $F_T \cap F_{T'}$, where $T < T'$, $T' \in \mathcal{S}^F_{>\emptyset}$, and such a cube contributes a simplex of dimension $\operatorname{Card}(T' - T) - 1$ to $Lk(v_T, F_T)$. The lemma follows. □

Consider the following condition on a Coxeter system (W, S).

(FC): *If $T \subseteq S$ and every pair of elements in T generates a finite subgroup of W, then T generates a finite subgroup of W.*

Lemma 4.3.4. *The following statements are equivalent.*

(1) $Lk(v_T, F_T)$ *is a flag complex for all* $T \in \mathcal{S}^f$.

(2) $\mathcal{S}^f_{>\emptyset}$ *is a flag complex.*

(3) (W, S) *satisfies* (FC).

Proof. If a simplicial complex is a flag complex, then so is the link of each of its simplices. Hence, using Lemma 4.3.3, we see that (1) and (2) are equivalent. The equivalence of (2) and (3) is immediate. □

Theorem 4.3.5. *Let X stand for K, Σ or Φ with the piecewise Euclidean, cubical structure described above, and let $d_\square$ be the resulting intrinsic metric on X. Then $(X, d_\square)$ is a $CAT(0)$-space if and only if (W, S) satisfies* (FC).

Proof. The spaces K, Σ and Φ are simply connected (by Theorem 1.5.1). So, by Corollary 4.2.3, $(X, d_\square)$ is a $CAT(0)$-space if and only if the link of each vertex in X is a flag complex. Hence, the theorem follows from Lemmas 4.3.1, 4.3.2 (to be proved in §4.5) and 4.3.4. □

Since a $CAT(0)$-space is contractible, we have, as a corollary, the following theorem, which was the main goal of this section.

Theorem 4.3.6. *If (W, S) satisfies* (FC), *then Φ is contractible.*

This theorem, together with Corollary 2.2.5 and Corollary 3.2.6, proves Theorem A of the Introduction.

Remark. Suppose that (FC) fails for (W, S). Let T be a minimal subset of S such that (FC) fails for (W_T, T). Then W_T is infinite, $\mathrm{Card}(T) \geq 3$, and every proper subset of T generates a finite Coxeter group. Given the list of diagrams of finite Coxeter groups (e.g. [Bo, Ch. VI, §4,.1, Théorème 1]), it is a simple matter to write down the diagrams of all such W_T, and this was done by Lannér in [La]. His well-known lists are repeated in Tables 1 and 2 on the next page. It turns out that any such W_T can be realized as a group generated by isometric reflections on Euclidean space or hyperbolic space of dimension $\mathrm{Card}(T) - 1$.

The previous remark proves the following.

Lemma 4.3.7. *(W, S) satisfies* (FC) *if and only if for all $T \subseteq S$, the diagram of (W_T, T) does not appear in Table* 1 *or* 2.

Remark. Theorem 4.3.6 also can be proved by a simple inductive argument, without showing that it admits a $CAT(0)$ metric. Indeed, if W is finite, then Φ is a cone and, therefore, contractible. If W is infinite and satisfies (FC), then there exist elements t_1, t_2 in S so that $t_1 t_2$ is of infinite order in W. Let $T_1 = S - \{t_1\}$, $T_2 = S - \{t_2\}$ and $T_0 = S - \{t_1, t_2\}$. Then $K = K_{T_1} \cup K_{T_2}$ and $K_{T_1} \cap K_{T_2} = K_{T_0}$, where K_{T_i} denotes $|(\mathscr{S}^f_{\leq T_i})'|$, $i = 0, 1, 2$. Each component of the inverse image of K_{T_i} in Φ is isomorphic to Φ_{T_i}. Since A is the amalgamated free product of A_{T_1} and A_{T_2} along A_{T_0}, the intersection of a component of type Φ_{T_1} and one of type Φ_{T_2} is either empty or isomorphic to Φ_{T_0}, and the nerve of the covering of Φ by these components is the tree associated to the amalgamated product. By induction, Φ_{T_i}, $i = 0, 1, 2$, is contractible, and the theorem follows.

(4.4) **The Moussong metric.** In this subsection we describe a different piecewise Euclidean structure on $X = K, \Sigma$ or Φ. Combinatorially, the cell structure on X will be the same as before: the cells are the dual faces, and they are combinatorial cubes. However, each such combinatorial cube will be identified with a different convex cell in Euclidean space (called a "Coxeter block", below). The resulting intrinsic metric on X will be denoted by d_M.

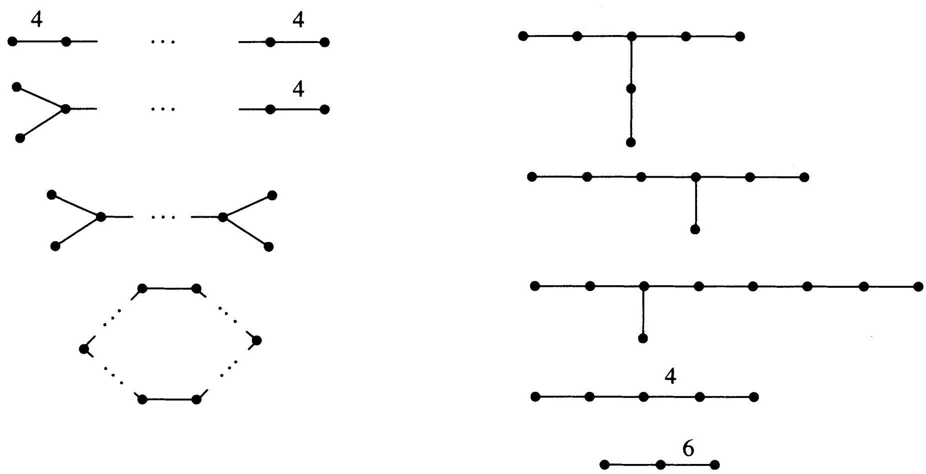

TABLE 1. Euclidean Coxeter groups with fundamental chamber an n-simplex , $n \geq 2$

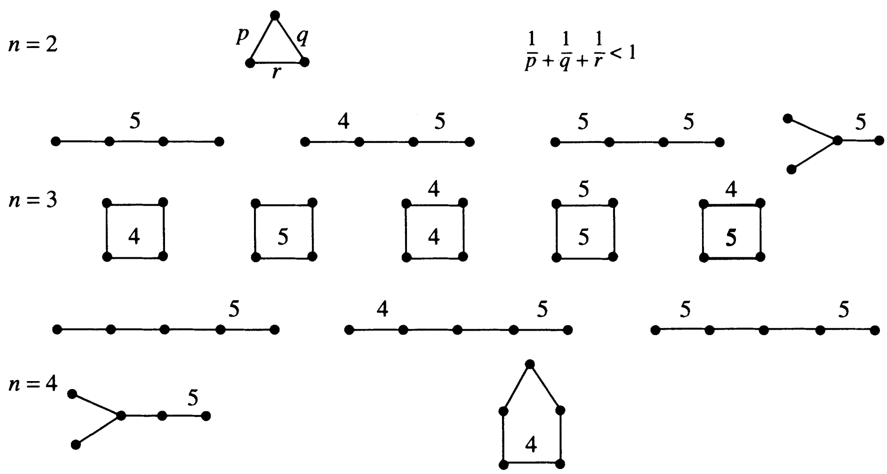

TABLE 2. Hyperbolic Coxeter groups with fundamental chamber an n-simplex

By Lemma 4.3.1, there is a simplicial isomorphism

$$Lk(v_T, X) \cong \hat{X}_T * Lk(v_T, F_T),$$

where $\hat{X}_T$ stands for Δ_T, $\hat{\Sigma}_T$ of $\hat{\Phi}_T$. In fact, this isomorphism will be an isometry if $*$ is interpreted as denoting the "orthogonal join" of two piecewise spherical complexes (as defined, for example, in the Appendix of [CD1]). Moreover, the metric d_M is designed so that the Coxeter complex $\hat{\Sigma}_T$, with the induced piecewise spherical structure, will be isometric to the round sphere $\mathbb{S}^{\mathrm{Card}(T)-1}$.

Suppose that (W, S) is a Coxeter system, that $S = \{s_1, \cdots, s_n\}$, and that (m_{ij}) is the associated Coxeter matrix. The associated *cosine matrix* (c_{ij}) is defined by $c_{ij} = -\cos(\pi/m_{ij})$ (where π/∞ is interpreted to be 0).

Suppose (c_{ij}) is positive definite. Choose a basis $\{u_i\}$ for $\mathbb{R}^n$, $n = \operatorname{Card}(S)$, so that $u_i \cdot u_j = c_{ij}$. We can then realize W as a linear reflection group on $\mathbb{R}^n$ by letting s_i act as the orthogonal reflection across the hyperplane $(u_i)^\perp$. A fundamental chamber is the simplicial cone

$$C_S = \{x \in \mathbb{R}^n | x \cdot u_i \leq 0, \forall i\}.$$

It follows that W is a discrete subgroup of $O(n)$, and hence finite. Conversely, if W is finite, then by representing it as a linear reflection group and choosing an invariant inner product we see that its cosine matrix must be positive definite. Thus, W is finite if and only if (c_{ij}) is positive definite (see [Bo, Ch. V, §4.8, Théorème 2]). The Coxeter complex is then naturally identified with $\mathbb{S}^{n-1}$, triangulated by the W-translates of the spherical simplex $\sigma_S = C_S \cap \mathbb{S}^{n-1}$.

Let $\{v_i\}$ denote the dual basis to $\{u_i\}$ and let C_S^* be the dual simplicial cone,

$$C_S^* = \{x \in \mathbb{R}^n | x \cdot v_i \geq 0, \forall i\}.$$

Equivalently, C_S^* is the cone of all nonnegative linear combinations of the u_i. Let σ_S^* be the dual spherical simplex,

$$\sigma_S^* = C_S^* \cap \mathbb{S}^{n-1}.$$

The vertex set of σ_S^* is $\{u_i\}$, and the length of the edge from u_i to u_j is $\cos^{-1}(c_{ij}) = \pi - \pi/m_{ij}$. Hence, σ_S^* has size $\geq \pi/2$ in the sense of §4.2.

Let $x_0 \in C_S$ be the point defined by $x_0 \cdot u_i = -1$, $i = 1, \cdots n$. The *Coxeter block* for (W, S) is the Euclidean polytope

$$B_S = C_S \cap (x_0 + C_S^*).$$

Its faces are the subsets B_{T_1, T_2}, for $T_1 \subseteq T_2 \subseteq S$, defined by

$$B_{T_1, T_2} = \{x \in B_S | x \cdot u_i = 0 \text{ for } s_i \in T_1 \text{ and } (x - x_0) \cdot v_j = 0 \text{ for } s_j \notin T_2\},$$

which is a convex polytope of dimension $\operatorname{Card}(T_2 - T_1)$. It follows that B_S is combinatorially equivalent to an n-cube. For $T \subseteq S$, the Coxeter block B_T can be naturally identified with the face $B_{\emptyset, T}$ of B_S.

For $T \subseteq S$, put $x_T = B_{T,T}$, so that $\{x_T\}$ is the vertex set of B_S. The faces $B_{T,S}$ and $B_{\emptyset, T}$ are contained in orthogonal faces of C_S and $x_0 + C_S^*$, respectively. Hence,

$$Lk(x_T, B_S) = Lk(x_T, B_{\emptyset, T}) * Lk(x_T, B_{T,S}) = \sigma_T * \sigma_{S-T}^*$$

(where $*$ denotes the orthogonal join). In particular, $Lk(x_S, B_S) = \sigma_S$ and $Lk(x_\emptyset, B_S) = \sigma_S^*$.

Now we return to the situation in which (W, S) can be infinite. As in §4.3, we view the fundamental chamber K as a union of combinatorial cubes F_T^*, $T \in \mathcal{S}^f$. Following [M] (or [CD2, §6]), we define a piecewise Euclidean structure on K by identifying F_T^* with B_T in such a fashion that for $T_1 \leq$

$T_2 \leq T$ the face $F_{T_1} \cap F^*_{T_2}$ of F^*_T is identified with B_{T_1, T_2}. In particular, the vertex v_T $(= F_T \cap F^*_T)$ is identified with x_T. Similarly, we get piecewise Euclidean structures on Σ and Φ.

Next we consider the induced piecewise spherical structure on links. Recall that, as an abstract simplicial complex, $Lk(v_\varnothing, F_\varnothing) = \mathcal{S}^f_{>\varnothing}$. Let $K_0 = |\mathcal{S}^f_{>\varnothing}|$, with the piecewise spherical structure obtained by identifying the simplex corresponding to T with the spherical simplex σ^*_T. As was previously noted, each such simplex has size $\geq \pi/2$. Also, $T \in \mathcal{S}^f_{>\varnothing} \Leftrightarrow W_T$ is finite $\Leftrightarrow$ the cosine matrix for W_T is positive definite $\Leftrightarrow \{\pi - \pi/m_{ij}\}_{s_i, s_j \in T}$ is the set of edge lengths of a spherical simplex. Hence, K_0 is a metric flag complex. Since the property of being a metric flag complex is inherited by links (cf., [M] or [CD 2, Lemma 2.41]) we get the following lemma as a corollary to Proposition 4.2.4.

Lemma 4.4.1 (Moussong [M]). *K_0 and the links of all simplices in K_0 are $CAT(1)$-spaces.*

Definition. Suppose W_T is finite. The *natural* piecewise spherical structures on the Coxeter complex $\hat{\Sigma}_T$ and the Deligne complex $\hat{\Phi}_T$ are defined by identifying the fundamental simplex Δ_T with the spherical simplex σ_T $(= C_T \cap \mathbb{S}^{\mathrm{Card}(T)-1})$.

Arguing as in Lemmas 4.3.1 and 4.3.3, we get the following.

Lemma 4.4.2. *Let $X = K, \Sigma$ or Φ with the piecewise Euclidean structure of Moussong discussed above. For $T \in \mathcal{S}^f$, let $\hat{X}_T = \sigma_T, \hat{\Sigma}_T$ or $\hat{\Phi}_T$ (as $X = K, \Sigma$ or Φ) with its natural piecewise spherical structure. Then the link of any vertex of X can be decomposed as an orthogonal join as follows:*

$$Lk(v_T, X) = \hat{X}_T * Lk(v_T, F_T).$$

Moreover, metrically, $Lk(v_\varnothing, F_\varnothing) = K_0$ and, for $T \neq \varnothing$, $Lk(v_T, F_T) = Lk(\sigma^*_T, K_0)$.

The orthogonal join of two piecewise spherical polyhedra is $CAT(1)$ if and only if each of them is $CAT(1)$ (by Lemmas A2 and A6 of the Appendix of [CD 1]). Hence, it only remains to decide when $\hat{X}_T$ is a $CAT(1)$-space. Since $\hat{\Sigma}_T$ is isometric to the round sphere, it is $CAT(1)$. This gives the following result of [M].

Theorem 4.4.3. (Moussong). *(Σ, d_M) is is $CAT(0)$.*

Conjecture 4.4.4. *(Φ, d_M) is $CAT(0)$.*

As we have seen, this conjecture is equivalent to Conjecture 3 of the Introduction, which we restate here.

Conjecture 3. *For any finite Coxeter system (W_T, T), the Deligne complex $\hat{\Phi}_T$ with its natural piecewise spherical structure is $CAT(1)$.*

Conjecture 3 is true when W_T is right-angled (i.e., when $W_T = (\mathbb{Z}/2)^T$), for in this case the natural piecewise spherical structure on $\hat{\Phi}_T$ coincides with the all right structure of §4.3. There is one other case when we know the conjecture is true.

Proposition 4.4.5. *Conjecture* 3 *is true when* $\mathrm{Card}(T) = 2$ (*so that* W_T *is the dihedral group* D_{2m} *of order* $2m$).

Proof. In this case $\hat{\Phi}_T$ is a 1-dimensional complex with all edges of length π/m. Hence, it is $CAT(1)$ if and only if it has no circuits with less than $2m$ edges. This condition holds by [AS, Lemma 6, p. 210]. $\square$

Thus, Conjecture 4.4.3 holds for W such that $\mathrm{Card}(T) \leq 2$ for all $T \in \mathcal{S}^f$ (for example, if W is of "large type" as defined in [AS]). Theorem B of the Introduction is an immediate consequence.

(4.5) **The Proof of Lemma 4.3.2.** Throughout this subsection, A *is an Artin group of finite type.* Our goal is to prove that the Deligne complex $\hat{\Phi}$ associated to A is a flag complex.

Before proving this we need a number of preliminary lemmas about Artin groups of finite type. These are based on work in [C1] and [C2], which in turn is based on ideas of Deligne [D] and Garside [Ga].

Let $F^+(S)$ be the free monoid on S and let A^+ be the quotient of $F^+(S)$ by the equivalence relation generated by $\mathrm{prod}(s_i, s_j; m_{ij}) = \mathrm{prod}(s_j, s_i, m_{ij})$, for $s_i, s_j \in S$. In other words, if $\langle S|R\rangle$ is the standard presentation for A, then A^+ is the monoid with monoid presentation $\langle S|R\rangle$. For example, when A is the braid group, A^+ is the monoid of positive braids.

The natural map $A^+ \to A$ is an injection; hence, we can identify A^+ with a subset of A. Moreover, for any $a \in A$, we can write $a = bc^{-1}$ for some $b, c \in A^+$. If $T \subseteq S$, then the natural map $A_T^+ \to A^+$ is also an injection (since $A_T \to A$ is), so we can also view A_T^+ as a subset of A.

Lemma 4.5.1. (i) *Suppose that* a, b, c *are in* A^+, *that* $a = bc$, *and that* $a \in A_T^+$. *Then* b *and* c *both lie in* A_T^+.

(ii) $A_T^+ = A_T \cap A^+$.

Proof. (i) Choose representative words $\hat{a} \in F^+(T)$ and $\hat{b}, \hat{c} \in F^+(S)$ for a, b and c. If $a = bc$ in A^+, then $\hat{a}$ is related to $\hat{b}\hat{c}$ via the equivalence relation generated by

$$\mathrm{prod}(s_i, s_j; m_{ij}) = \mathrm{prod}(s_j, s_i; m_{ij}).$$

Under this relation, no generator can be completely eliminated. Hence, if $\hat{b}$ or $\hat{c}$ involves some $s_i \notin T$, then so does $\hat{a}$.

(ii) Clearly, $A_T^+ \subset A_T \cap A^+$. Conversely, let $a \in A_T \cap A^+$. Write $a = bc^{-1}$ for $b, c \in A_T^+$. Then $ac = b$ in A^+, so by part (i), a lies in A_T^+. $\square$

There are two partial orderings on A^+, denoted $<_\ell$ and $<_r$, and defined by:

$$a <_\ell b, \quad \text{if } ac = b \text{ for some } c \in A^+,$$

$$a <_r b, \quad \text{if } ca = b \text{ for some } c \in A^+.$$

Recall the section $m : W \to A$ defined in §1.1. Let w_ℓ denote the (unique) element of longest length in W, and let $\Delta = m(w_\ell)$. Then Δ^2 is central in A and, for any $a \in A$, $\Delta^k a$ and $a\Delta^k$ lie in A^+ for k sufficiently large.

Let $M = m(W)$. By definition of m, $M \subset A^+$. In fact, we can describe M in terms of the partial orderings on A^+ as follows:

$$M = \{a \in A^+ | a <_\ell \Delta\} = \{a \in A^+ | a <_r \Delta\}.$$

Since $S \subset M$, M generates A^+ as a monoid. If M^k denotes the set of elements in A^+ which can be written as a product of at most k elements of M, then it follows from [C2, Remark 2.4 and Lemma 3.4] that

$$M^k = \{a \in A^+ | a <_\ell \Delta^k\} = \{a \in A^+ | a <_r \Delta^k\}.$$

Lemma 4.5.2. *Let $a, b \in A^+$ and let $* = \ell$ or r.*

(i) *$\{c \in A^+ | c <_* a$ and $c <_* b\}$ has a unique maximal element, denoted* $\mathrm{glb}_*(a, b)$.

(ii) *$\{c \in A^+ | a <_* c$ and $b <_* c\}$ has a unique minimal element, denoted* $\mathrm{lub}_*(a, b)$.

(iii) *If $T \subseteq S$, then $\{c \in A^+ | c <_* a\} \cap A_T^+$ has a unique maximal element, denoted* $\mathrm{glb}_*(a, A_T^+)$.

Proof. By symmetry, it suffices to prove each part of the lemma for either $* = \ell$ or $* = r$.

(i) For $* = r$, this is precisely Corollary 2.7 of [C2].

(ii) Choose k large enough so that $a, b \in M^k$, hence $a <_\ell \Delta^k$ and $b <_\ell \Delta^k$. Let $a_k = a^{-1}\Delta^k$ and $b_k = b^{-1}\Delta^k$. Then $a_k \in A^+$ and $a_k <_r \Delta^k$, and similarly for b_k. Consider the posets

$$L_k = \{c \in A^+ | a <_\ell c, b <_\ell c, c <_\ell \Delta^k\}, \qquad R_k = \{c \in A^+ | c <_r a_k, c <_r b_k\},$$

with L_k partially ordered by $<_\ell$ and R_k partially ordered by $<_r$. The map $c \to c_k = c^{-1}\Delta^k$ is an order-reversing bijection from L_k to R_k. Thus, if d_k is the unique maximal element of R_k (which exists by part (i)), then $\hat{d}_k = \Delta^k d_k^{-1}$ is the unique minimal element of L_k. Finally, note that since $a_{k+1} = a_k\Delta$ and $b_{k+1} = b_k\Delta$, we must have $\hat{d}_{k+1} = \Delta^{k+1} d_{k+1}^{-1} = \Delta^k d_k^{-1} = \hat{d}_k$. In other words, the sets $L_k \subseteq L_{k+1} \subseteq L_{k+2} \subseteq \cdots$ all have the same unique minimal element, $\hat{d}_k$. Thus, $\hat{d}_k$ is also the unique minimal element for

$$\bigcup_{j \geq k} L_j = \{c \in A^+ | a <_\ell c, b <_\ell c\},$$

i.e., $\hat{d}_k = \mathrm{lub}_\ell(a, b)$.

(iii) Let $M_T = m(W_T) = M \cap A_T^+$ and let Δ_T be the image of the longest element of W_T. It follows from Lemma 4.5.1(i) that $M^k \cap A_T^+ = (M_T)^k$. Choosing k so that $a <_* \Delta^k$, we therefore have that

$$\begin{aligned}\{c \in A^+ | c <_* a\} \cap A_T^+ &= \{c \in A^+ | c <_* a\} \cap M^k \cap A_T^+ \\ &= \{c \in A^+ | c <_* a\} \cap M_T^k = \{c \in A^+ | c <_* a, c < \Delta_T^k\},\end{aligned}$$

which has a unique maximal element by part (i). $\square$

Lemma 4.5.3. *Let $a \in A$ and $T \subseteq S$. If $aA_T \cap A^+ \neq \varnothing$, then it contains a unique minimal element a_0 (with respect to $<_\ell$), and $aA_T \cap A^+ = a_0 A_T^+$.*

Proof. Since $a <_\ell b$ and $a \neq b$ implies that the word length of a (with respect to S) is strictly less than the word length of b, there are no infinite descending chains in A^+. Hence, if $aA_T \cap A^+$ is nonempty, it must have minimal elements.

Suppose a_1 and a_2 are both minimal in $aA_T \cap A^+$. Then $a_1^{-1}a_2 \in A_T$, so we can write $a_1^{-1}a_2 = pq^{-1}$ for $p, q \in A_T^+$. Thus,

$$a_1 p = a_2 q, \quad p, q \in A_T^+.$$

Let $c_0 = \mathrm{lub}_r(p, q)$ in A_T^+. Then $a_1 p = a_2 q = a_0 c_0$ for some $a_0 \in A^+$. It follows that $a_0 A_T = a_1 A_T = a_2 A_T = aA_T$, i.e., $a_0 \in aA_T \cap A^+$. On the other hand, $p <_r c_0$, so $c_0 = dp$ for some $d \in A^+$. Thus, $a_0 dp = a_0 c_0 = a_1 p$, so $a_0 d = a_1$. In other words, $a_0 <_\ell a_1$. The same argument applied to $q <_r c_0$ shows that $a_0 <_\ell a_2$. But this contradicts the minimality of a_1 and a_2 unless $a_0 = a_1 = a_2$. Finally note that if b is any element of $aA_T \cap A^+$, then $a_0 <_\ell b$, so by Lemma 4.5.1(ii), $a_0^{-1}b \in A_T \cap A^+ = A_T^+$, i.e., $aA_T \cap A^+ = a_0 A_T^+$. □

Recall that a "special subgroup" of A is a subgroup of the form A_T for some $T \subseteq S$. It follows from [D, Theorem 4.14] that the intersection of two special subgroups is a special subgroup.

Lemma 4.5.4. *Suppose, A_1, A_2 and $A_{12} = A_1 \cap A_2$ are special subgroups of A and that $g \in A$ is such that $gA_{12} \cap A^+ \neq \varnothing$. If a_1 and a_2 are the minimum elements of $gA_1 \cap A^+$ and $gA_2 \cap A^+$, respectively (minimal with respect to $<_\ell$), then the minimum element of $gA_{12} \cap A^+$ is $a = lub_\ell(a_1, a_2)$.*

Proof. By Lemma 4.5.3, $gA_{12} \cap A^+ = gA_1 \cap gA_2 \cap A^+ = a_1 A_1^+ \cap a_2 A_2^+$. Let b be the minimum element of $gA_{12} \cap A^+$. Then $b = a_1 c_1 = a_2 c_2$ for some $c_1 \in A_1^+$, $c_2 \in A_2^+$. Thus, b is an upper bound for a_1 and a_2, so $a <_\ell b$. Write $b = ac$. Then $c <_r c_1$ and $c <_r c_2$, so by Lemma 4.5.1(i), $c \in A_1^+ \cap A_2^+ = A_{12}^+$. It follows that $a \in gA_{12} \cap A^+$. This contradicts the minimality of b unless $a = b$. □

Proof of Lemma 4.3.2 (that $\hat{\Phi}$ is a flag complex). The poset of simplices of $\hat{\Phi}$ is isomorphic to the poset of special cosets aA_T, for $T \subsetneqq S$, under reverse inclusion. In particular, a k-simplex σ of $\hat{\Phi}$ corresponds to a coset aA_T with $\mathrm{Card}(S - T) = k + 1$ and $Lk(\sigma, \hat{\Phi}) = \hat{\Phi}_T$.

Suppose $\hat{\Phi}$ is not a flag complex. Then there exists a set of vertices $\{v_1, \ldots, v_k\}$, $k \geq 3$, of $\hat{\Phi}$ which are pairwise joined by edges, but do not span a simplex of $\hat{\Phi}$. Choosing a minimal such set of vertices, we may assume that every proper subset of $\{v_1, \ldots, v_k\}$ spans a simplex. In this case, we say that $\{v_1, \ldots, v_k\}$ is an "empty simplex" in $\hat{\Phi}$. It suffices to consider the case $k = 3$; for if $k > 3$ and σ is the span of $\{v_4, \ldots, v_k\}$, then $\{v_1, v_2, v_3\}$ is an "empty triangle" in $Lk(\sigma, \hat{\Phi}) = \hat{\Phi}_T$.

A triangle in $\hat{\Phi}$ corresponds to a diagram of special cosets

$$\begin{tikzcd}[column sep=small]
a_1A_1 \arrow[rr, phantom, "\supset"] \arrow[dr, phantom, sloped, "\supset"] & & a_{12}A_{12} \arrow[rr, phantom, "\subset"] & & a_2A_2 \arrow[dl, phantom, sloped, "\supset"] \\
& a_{13}A_{13} \arrow[dr, phantom, sloped, "\subset"] & & a_{23}A_{23} & \\
& & a_3A_3 \arrow[ur, phantom, sloped, "\supset"] & &
\end{tikzcd}$$

where $a_{ij}A_{ij} = a_iA_i \cap a_jA_j$. To show that this triangle is not "empty" we must show that $a_1A_1 \cap a_2A_2 \cap a_3A_3 \neq \varnothing$. After translating by a sufficiently high power of Δ, we may assume that all cosets in the above triangle intersect A^+. We can then choose each coset representative a_i to be the minimum element of $a_iA_i \cap A^+$ (by Lemma 4.5.3), and choose a_{ij} to be $\mathrm{lub}_\ell(a_i, a_j)$, which is the minimum element of $a_{ij}A_{ij} \cap A^+$ (by Lemma 4.5.4). Then $a_{12} \in a_1A_1^+$, so $a_{12} = a_1p$ for some $p \in A_1^+$. Similarly, $a_{13} = a_1q$ for some $q \in A_1^+$. Set

$$a = \mathrm{lub}_\ell(a_1, a_2, a_3) = \mathrm{lub}_\ell(a_{12}, a_{13}) = a_1\mathrm{lub}_\ell(p, q).$$

Then $a \in a_1A_1^+$. By symmetry, we must also have $a \in a_2A_2^+$ and $a \in a_3A_3^+$. Thus, $a_1A_1^+ \cap a_2A_2^+ \cap a_3A_3^+$ contains the element a, and hence cannot be empty. $\square$

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