SEMIGROUPS OF PARTIAL ISOMETRIES

BY LAWRENCE J. WALLEN

Communicated by Paul Halmos, February 24, 1969

Let $\{S_t\}$ be the (strongly continuous) semigroup of operators on $L_2(0, 1)$ defined as follows: if $0 \leqq t < 1$, then

$$S_t f(x) = 0 \qquad \text{if } x \leqq t,$$
$$= f(x-t) \qquad \text{if } t < x \leqq 1;$$

if $t \geqq 1$, then $S_t = 0$. The operators S_t are related to the classical Volterra operator J $(Jf(x) = \int_0^x f(t)dt)$ by the equation $J = \int_0^1 S_t dt$ or, what comes to the same thing, $i\lambda J(I - i\lambda J)^{-1} = \int_0^1 e^{i\lambda t} S_t dt$. These formulas, together with the uniqueness of the Fourier transform, permit one to pass readily from considerations concerning J to those about $\{S_t\}$ and vice versa. This correspondence was used by Dixmier in [1] where, however, he considers these operators on L_1.

Let us note three properties of $\{S_t\}$:

(a) each S_t is a partial isometry,

(b) $S_1 = 0$ and $S_t \neq 0$ if $0 \leqq t < 1$,

(c) $\{S_t\}$ is irreducible.

It turns out that (a), (b), and (c) characterize $\{S_t\}$ up to unitary equivalence. This is a special case of the following result.

THEOREM. *If $\{W_t\}$ is a strongly continuous semigroup of operators on a Hilbert space and if $\{W_t\}$ satisfies* (a) *and* (b), *then a necessary and sufficient condition that $\{W_t\}$ be unitarily equivalent to a direct sum of n copies of $\{S_t\}$ (n may be infinite) is that the von Neumann algebra generated by $\{W_t\}$ be a factor.*

The detailed proof will be published elsewhere, but a sketch may be of interest. The basic fact used (see [2]), is that a nilpotent partial isometry, all of whose powers are partial isometries has a sort of Jordan decomposition. Such an operator is, in fact, the direct sum of operators having a matrix representation:

$$\begin{pmatrix} 0 & 0 & \cdots & & 0 \\ I & 0 & \cdots & & 0 \\ 0 & I & \cdots & & 0 \\ \vdots & & & & \vdots \\ 0 & & & I & 0 \end{pmatrix}$$

A careful analysis of this representation applied to W_{2-k} permits us to construct projections E_ξ, $\xi \geqq 0$, in the center of the von Neumann algebra generated by $\{W_t\}$, with the property that $W_t|\text{ran } E_\xi = 0$ if $t \geqq \xi$. The assumption that the algebra is a factor enables us to conclude that for $0 \leqq t \leqq 1$, $\ker(W_t) = \text{ran}(W_{1-t})$. It then follows easily that the minimal isometric dilation $\{V_t\}$ has no unitary part and that $H = \text{ran}^{\perp}(V_1)$. Finally, the theorem is established by appealing to Cooper's characterization of isometric flows (see [3]).

References

1. J. Dixmier, *Les operateurs permutable a l'operateurs integral*, Portugal. Math. **8** (1949), 73–84.

2. P. R. Halmos and L. J. Wallen, *Powers of parital isometries*, J. Math. Mech. (to appear).

3. J. L. B. Cooper, *One-parameter semi-groups of isometric operators in Hilbert space*, Ann. of Math. **48** (1947), 827–842.

University of Hawaii, Honolulu, Hawaii 96822