UPPER BOUNDS FOR PERMANENTS OF (0, 1)-MATRICES

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1. **Introduction.** A matrix is said to be a (0, 1)-*matrix* if each of its entries is either 0 or 1. If $A=(a_{ij})$ is an n-square matrix then the *permanent* of A is defined by

$$p(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n. Bounds for permanents of general (0, 1)-matrices and for permanents of certain subclasses of (0, 1)-matrices are of combinatorial significance and yet virtually the only known upper bound for $p(A)$ is the obvious one, the product of row sums of A. It has been conjectured that the permanent of an n-square (0, 1)-matrix with exactly k ones, $k<n$, in each row and column must exceed $n!(k/n)^n$ [1, p. 59]. It has been also conjectured by H. J. Ryser that in the class of all mk-square (0, 1)-matrices with exactly k ones in each row and column the maximum permanent is equal to $(k!)^m$, i.e., to the permanent of the direct sum of k-square matrices all of whose entries are 1. In the present note I give a significant upper bound for the permanent of a general (0, 1)-matrix. I also conjecture an upper bound which would allow one to answer Ryser's conjecture in the affirmative.

2. **Results.**

LEMMA. *If $r_1, \cdots, r_c$ are positive integers then*

$$\sum_{j=1}^{c} \frac{2}{r_j} \prod_{t=1}^{c} \frac{r_t}{r_t+1} \leqq 1$$

with equality if and only if $c \leqq 2$ and either r_1 or r_2 is equal to 1.

PROOF. Let E_s denote the sth elementary symmetric function of the numbers $1/r_1, \cdots, 1/r_c$; then

$$(1) \qquad 0 \leqq \prod_{t=1}^{c} (1 - 1/r_t) = 1 - E_1 + E_2 - E_3 + \cdots + (-1)^c E_c$$

with equality if and only if one of the r_t is 1. Therefore

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$$1 + E_1 + E_2 - E_3 + \cdots + (-1)^c E_c \geqq 2E_1$$

and a fortiori

$$\prod_{t=1}^{c} (1 + 1/r_t) \geqq 2E_1,$$

i.e.,

$$1 \geqq \sum_{j=1}^{c} \frac{2}{r_j} \prod_{t=1}^{c} \frac{r_t}{r_t + 1}$$

and the inequality is strict unless (1) is an equality and $E_3 = 0$, i.e. $c \leqq 2$.

THEOREM. *Let* $A = (a_{ij})$ *be an* n*-square* $(0, 1)$*-matrix and let* $r_i = \sum_{j=1}^{n} a_{ij}$, $i = 1, \cdots, n$; *then*

$$p(A) \leqq \prod_{i=1}^{n} \frac{r_i + 1}{2} \tag{2}$$

with equality if and only if A *is a permutation matrix.*

PROOF. We use induction on n. Since the permanent is unchanged if the rows of the matrix are permuted we can assume that $a_{i1} = 1$, $i = 1, \cdots, c$, and $a_{i1} = 0$, $i = c+1, \cdots, n$. Denote the submatrix obtained from A by deleting the ith row and the jth column by $A(i|j)$. Then, by the induction hypothesis,

$$\begin{aligned} p(A(i|1)) &\leqq \left(\prod_{\substack{t=1 \\ t \neq i}}^{c} \frac{r_t}{2} \right) \left(\prod_{j=c+1}^{n} \frac{r_j + 1}{2} \right) \\ &= \frac{2}{r_i} \left(\prod_{t=1}^{c} \frac{r_t}{r_t + 1} \right) \left(\prod_{j=1}^{n} \frac{r_j + 1}{2} \right), \qquad i = 1, \cdots, c, \end{aligned}$$

with equality if and only if $A(i|1)$ is a permutation matrix. Thus expanding $p(A)$ by the elements of the first column

$$\begin{aligned} p(A) &= \sum_{i=1}^{c} p(A(i|1)) \\ &\leqq \sum_{i=1}^{c} \frac{2}{r_i} \left(\prod_{t=1}^{c} \frac{r_t}{r_t + 1} \right) \left(\prod_{j=1}^{n} \frac{r_j + 1}{2} \right) \\ &\leqq \prod_{j=1}^{n} \frac{r_j + 1}{2}, \text{ by the lemma.} \end{aligned}$$

Equality holds in (2) if and only if $A(i|1)$ are permutation matrices,

$i=1, \cdots, c$, and $c \leqq 2$, r_1 or $r_2=1$. But this implies that A is a permutation matrix.

CONJECTURE. *If $A=(a_{ij})$ is an n-square* (0, 1)*-matrix then*

$$p(A) \leqq \prod_{i=1}^{n} (r_i!)^{1/r_i} \tag{3}$$

with equality if and only if there exist permutation matrices P and Q such that PAQ is a direct sum of matrices all of whose entries are 1.

The conjecture is known to be true for all (0, 1)-matrices whose row sums do not exceed 6.

REFERENCE

1. H. J. Ryser, *Combinatorial mathematics*, Carus Math. Monograph No. 14, Math. Assoc. Amer., 1963.

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THE COLLINEATION GROUPS OF DIVISION RING PLANES. I. JORDAN ALGEBRAS

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In this note, we outline a method which reduces the determination of the collineation group of a division ring plane to the solution of certain algebraic problems—in particular, to the question of when two rings of a certain type are isomorphic. This method is then applied to planes coordinatized by finite dimensional Jordan algebras of characteristic $\neq 2, 3$, and their collineation groups are determined. Complete arguments and detailed proofs will appear elsewhere.

1. Let $\mathfrak{R}$ be a nonalternative division ring, let $\pi(\mathfrak{R})$ be the projective plane coordinatized by $\mathfrak{R}$, and let $G(\pi)$ be the collineation group of π. Then (see [1]) $G(\pi)$ possesses a solvable normal subgroup whose structure is known, the elementary subgroup, such that the factor group is isomorphic with the group of *autotopisms* of $\mathfrak{R}$, $A(\mathfrak{R})$. Also, $A(\mathfrak{R}) \approx H(\pi)$, where $H(\pi)$ consists of those elements of $G(\pi)$ which leave fixed the points (∞), (0), and (0, 0). (See [2], Chapter 20 for the coordinatization of projective planes.)

Let $B(\mathfrak{R})$ be the *automorphism* group of $\mathfrak{R}$. Then $B(\mathfrak{R}) \approx H_1(\pi)$, where $H_1(\pi)$ consists of those elements of $H_1(\pi)$ which leave the point